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Asset Allocation with Nonnegative Weights and Lognormal Portfolio Returns

Luigi Buzzacchi[1] and Luca Ghezzi[2]

The stage of strategic asset allocation is the most important one in a process of portfolio management:asset classes are selected and target weights are set. Careful decision-making benefits from the computation of an efficient frontier. In this work, weights are nonnegative and rebalanced once a year, portfolio returns are time uncorrelated and lognormal. A novel sufficient condition is obtained, whereby efficient portfolios based on linear returns may turn into efficient portfolios based on logarithmic returns. If that is met, the efficient frontier based on logarithmic returns is upward sloping, stretching from a corner portfolio with global minimum-variance to acorner portfolio with global maximum-variance. Such a complementary efficient frontier allows a decision maker to forecast the long-term portfolio value. The null hypothesis of lognormal portfolio returns is also tested by using two different data sets. It is always rejected in the latter; it is either accepted or rejected in the former, depending on the specific efficient portfolio.

JEL codes: G11, C61, and C12

1. Introduction

As remarked by Sharpe et al. (2007) and Gibson (2008) among others, the stage of strategic asset allocation is the most important one in a process of portfolio management. Asset classes are selectedand target weights are set so as to achieve a suitable exposure to systematic risks.The stages of an investment process are recalled in Maginn et al. (2007) and, briefly, in Ghezzi (2018). Careful decision-making benefits from the computation of an efficient frontier, which displays the trade-off between risk and return.

In this paper,a single investment is considered with all coupons and dividends being reinvested. Target weights are supposed to be nonnegative and constant, whereas actual weights are subject to annual rebalancing. Portfolio returns are assumed to be time uncorrelated and lognormal. Therefore,

- multi-period optimization boils down to single period optimization;
- the long-term portfolio value depends on three variables: the sample mean of its logarithmic returns, the initial investment, and the time horizon.

As remarked by Sharpe et al. (2007) among others, forecasting the long-term portfolio value is an important step in a strategic asset allocation. Indeed, the best

[1]Full Professor of Industrial Economics, Interuniversity Department of Regional and Urban Studies and Planning & FULL, Politecnico di Torino, Viale Mattioli 39, 10125 Torino, Italy,
Email: luigi.buzzacchi@polito.it

[1]ScD, Associate Professor of Financial Modelling and Management, Department of Integrated Business Management, Università Carlo Cattaneo, Corso Matteotti 22, 21053 Castellanza (VA), Italy,
Email: lghezzi@liuc.it

possible forecast of a sample mean is a population mean. If the assumption of lognormal portfolio returns is tenable, a population mean of logarithmic returns is provided by our novel sufficient condition.

More precisely, linear quadratic optimization is complemented with a lognormal mapping, resulting in a rigorous and operationally useful procedure. A novel sufficient condition is obtained, whereby only efficient portfolios based on linear returns may turn into efficient portfolios based on logarithmic returns.Notably, if short sellingis assumed away, no other theoretical findings are available about the linear and logarithmic returns of efficient portfolios.

Two data sets are used by way of illustration under the tentative yet usual assumption that forecast moments are the same as the historical moments that are available at the decision date. As explained by Haugen (1997), such working assumption doesn’t apply to individual securities, since their past returns are affected by many idiosyncratic events that are unlikely to occur again.Our data sets include the annual total returns of three major US asset classes over the period 1872-2011 as well as the annual total returns of four equity classes over the period 1972-2017.

The efficient frontier based on logarithmic returns is upward sloping in both instances,stretching from a portfolio with global minimum-variance to a portfolio with global maximum-variance. Nonetheless, if short selling and hence negative weights were allowed, the efficient frontier based on logarithmic returns wouldn’t be upward sloping in both instances, a pattern highlighted by Ghezzi (2018).

The null hypothesis of lognormal portfolio returns is then testedto check whether our sufficient conditioncould be useful in practice. If linear returns are lognormal, logarithmic returns are normal and vice versa. Use is made of RStudio software.Seven different efficient portfolios are taken into consideration and four different statistical tests are performed. The above-mentioned null hypothesis is

- either accepted or rejected in the former data set, depending on the specific efficient portfolio. Nonetheless, it is rejected for each constituent asset class.
- is always rejected in the latter data set.

The paper is organised as follows.A literature review is presented in Section 2. Data sets are introduced in Section 3. Theoretical and empirical findings are presented in Section 4, whereasadditional empirical findings are reported in Section 5. Conclusions are provided in Section 6.

2. Literature Review

Optimization, simulation, and sensitivity analysismay beperformed in support of a strategic asset allocation. The procedures in use nowadays rest on the mean-variance approach, which was developed by Professor Harry Markowitz in the 1950s. Those procedures are reviewed by Sharpe et al. (2007); they also include the resampled efficient frontier and the Black-Litterman approach.

According to the mean-variance approach, each feasible asset allocation, i.e. each feasible set of weights, is represented by a mean return $\mu$and a standard deviation σ;

the former measures return, whereas the latter measures risk. First, linear quadratic optimization is carriedout and an efficient frontier is detected.Next, the most appropriate efficient portfolio is selected in line with the investor's risk tolerance.

Linear quadratic optimization may be unrestricted or restricted. Unrestricted optimization is the simplest since it only requiresthat the sum of weights is equal to one. Unfortunately, it may call for long-short asset allocations.

In Merton (1972) the efficient frontier is derived analytically, thus obtaining a parabola in $(\mu;\sigma^2)$ space as well as a hyperbola in $(\mu;\sigma)$ space, i.e.

$$\sigma^2 = \frac{a}{d}\mu^2 - \frac{2b}{d}\mu + \frac{c}{d} > 0 \text{ and } \sigma = \sqrt{\frac{a}{d}\mu^2 - \frac{2b}{d}\mu + \frac{c}{d}} \quad (1)$$

The four real parameters are

$$a = I^T\Sigma^{-1}I > 0; b = \mathrm{M}^T\Sigma^{-1}I; c = \mathrm{M}^T\Sigma^{-1}M > 0; d = ac - b^2 > 0 \quad (2)$$

where I is a unit vector, $\mathrm{M} = [\mu_j]$ is a mean vector, and $\Sigma = [\sigma_{jk}]$ is a variance-covariance matrix that is real, symmetric, and positive definite.Asset classes are represented by those means, variances, and covariances. As shown in Rudolf (1994),we have

$$W = \Sigma^{-1}\mathrm{M}\left(\frac{a\mu - b}{d}\right) - \Sigma^{-1}I\left(\frac{b\mu - c}{d}\right) \quad (3)$$

where $W = [w_j]$ is a vector including the weights of the efficient portfolio with mean μ and standard deviation σ given by Equation (1).

Sign-restricted optimization is more useful yet more challenging. It requires thatweights arenonnegative and sum to one so that asset allocations are only long. As stated inMarkowitz (1991), the efficient frontier is piecewise hyperbolic; as proved in Rudolf (1994), the minimum-variance set is piecewise hyperbolic, continuous, non-differentiable, and strictly concave. Each hyperbolic piece includesonly dominant asset classes, all having positive weights. In contrast, all dominated asset classes have zero weights. Each hyperbolic piece has two corner portfolios as bounds. When going through a corner portfolio, either a dominant asset class becomes dominated or vice versa or both do occur. The asset class with the highest mean is a corner portfolio.

The means,variances, and covariances of the linear returns of all asset classes are hard to estimate.Moreover, long-only asset allocations are very sensitive to small estimation errors. As reported by Sharpe et al. (2007), errors in means have an impact 10 times larger than errors in variances and 20 times larger than errors in covariances. Jorion (1992) is a clear example of how optimization and Monte Carlo simulation can be used together to analyse the effects exerted by estimation errors. Luenberger (2014) examines both the effects of estimation errors and the methods that can mitigate such effects. Altogether, the strong sensitivity to small estimation errors is troublesome, since it may go along with long-only asset allocations that have extreme weights. A heuristic remedy is to set part of an asset allocation in advance by placing lower and upper bounds on the weights of all asset classes (Gibson, 2008).

If intermediate investments or divestments as well as taxes are assumed away, incomplete theoretical findings are available about the link between an efficient frontier and a range of final portfolio values, namely about the link between the linear and logarithmic returns of efficient portfolios.

Ghezzi (2018) expands on Merton (1972) by assuming that portfolio returns are lognormally distributed. The discriminant

$$\Delta = (a + 3b - 2d)^2 - 4(a + d)(b + 2c + d) \tag{4}$$

is eventually determined by considering the one-to-one mapping between the space of linear moments $(\sigma; \mu)$ and the space of logarithmic moments $(s; m)$, i.e. the equations involving two pairs of population moments, reported in Crow and Shimizu (1988)

$$1 + \mu = e^{m+0.5s^2} \tag{5}$$

$$\sigma^2 = (1 + \mu)^2 \left(e^{s^2} - 1\right) \tag{6}$$

An efficient portfolio T is determined in the former space so that it maps into the minimum-variance portfolio in the latter space. According to Propositions 1, 2, and 3 in Ghezzi (2018)

- all efficient or minimum-variance portfolios below point T map into minimum-variance portfolios in the space $(s; m)$;
- if the discriminant (4) is negative, all efficient portfolios above point T map into efficient portfolios in the space $(s; m)$;
- if the discriminant (4) is positive, some efficient portfolios above point T map into inefficient portfolios in the space $(s; m)$.

This paper fills a theoretical gap in the scientific literature by considering long-only asset allocations under the assumption of lognormal portfolio returns.

3. Data Sets

Use is made of annual total returns. All coupons and dividends are assumed to be reinvested.

The formerdata setspans the historical period 1872-2011. It includes three major US asset classes: Treasury bills with a maturity of 1 year, Treasury bonds with a maturity of 10 years, and S&P Composite. Original data were downloaded in winter 2019 from the webpage of Professor Bob Shiller, Yale University. Actual data were obtained by assuming that

- the yield to maturity of T-bills is proxied by an appropriate interest rate;
- T-bonds are bought at par and held only for 1 year.

The latter data set was downloaded in summer 2018 from Bloomberg Finance LP. It covers the historical period 1972-2017. It includes four major equity classes: US stocks (S&P 500 index), international stocks (MSCI EAFE index), real estate securities (FTSE NAREIT Equity REITS index), and commodity-linked securities (S&P GSCI index). Remarkably, an international gold standard was abandoned between 1971 and 1973.

Annual linear returns are considered in Table 1. The sample moments of all three major US asset classes are reported in Table 1a, whereas the sample moments of all four equity classes are reported in Table 1b.

Table 1(a): Sample Moments of Three Major US Asset Classes: Annual Linear Returns, 1872-2011

	Mean return %	Standard deviation %	Correlation matrix		
			T-Bills	T-Bonds	S&P Composite
T-bills	4.70	2.80	1		
T-bonds	4.91	5.83	0.34	1	
S&P Composite	10.19	18.04	-0.10	0.07	1

Table 1(b): Sample Moments of Four Major Equity Classes: Annual Linear Returns, 1972-2017

	Mean return %	Standard deviation %	Correlation matrix			
			S&P 500	EAFE	NAREIT	GSCI
S&P 500	11.88	17.09	1			
EAFE	11.85	21.83	0.66	1		
NAREIT	13.37	17.75	0.54	0.38	1	
GSCI	9.66	25.43	-0.05	0.07	-0.05	1

Annual logarithmic returns are considered in Table 2. The sample moments of all three major US asset classes are reported in Table 2a, whereas the sample moments of all four equity classes are reported in Table 2b.

Table 2(a): Sample Moments of Three Major US Asset Classes: Annual Logarithmic Returns, 1872-2011

	Mean return %	Standard deviation %	Correlation matrix		
			T-Bills	T-Bonds	S&P Composite
T-bills	4.56	2.64	1		
T-bonds	4.65	5.31	0.32	1	
S&P Composite	8.30	17.14	-0.08	0.06	1

Table 2(b): Sample Moments of Four Major Equity Classes: Annual Logarithmic Returns, 1972-2017

	Mean return %	Standard deviation %	Correlation matrix			
			S&P 500	EAFE	NAREIT	GSCI
S&P 500	9.94	16.77	1			
EAFE	9.21	20.58	0.74	1		
NAREIT	11.20	17.15	0.59	0.48	1	
GSCI	6.32	25.06	0.06	0.19	0.07	1

Arithmetic, geometric, and logarithmic means fulfil two inequalities, reported in Ghezzi (2018). The numerical computation of geometric means is examined by Markowitz (1991).

The impact of regular rebalancing on a diversified portfolio is analysed by Booth and Fama (1992) and Willenbrock (2011), who put forward a breakdown of portfolio return; Booth and Fama (1992) focus on the logarithmic mean, whereas Willenbrock (2011) considers the geometric mean.

When drawing an unrestricted efficient frontier, we assume that our population moments are the same as the sample moments of either Table 1a or Table 1b (see Haugen 1997).

The statistical analysis of historical data is usual in finance; it can also deal with the performance of important asset classes over a very long term, as in Jorion and Goetzmann (1999) and Dimson et al. (2002). Indeed, a poor understanding of the historical record could go along with poor forecasts about the future. Notably, our larger data set includes 140 annual returns. It serves the purpose of inferential statistics; nonetheless, it couldn’t come in useful in business practice owing to a questionable predictive power.

4. Theoretical and Empirical Findings

Recall that a single investment is considered with all coupons and dividends being reinvested. Actual weights are nonnegative and subject to annual rebalancing.If portfolio returns are assumed to be time uncorrelated and lognormal, multi-period optimization is the same as single period optimization. Nonnegative weights within the above-mentioned setting haven’t been considered in the scientific literature yet.

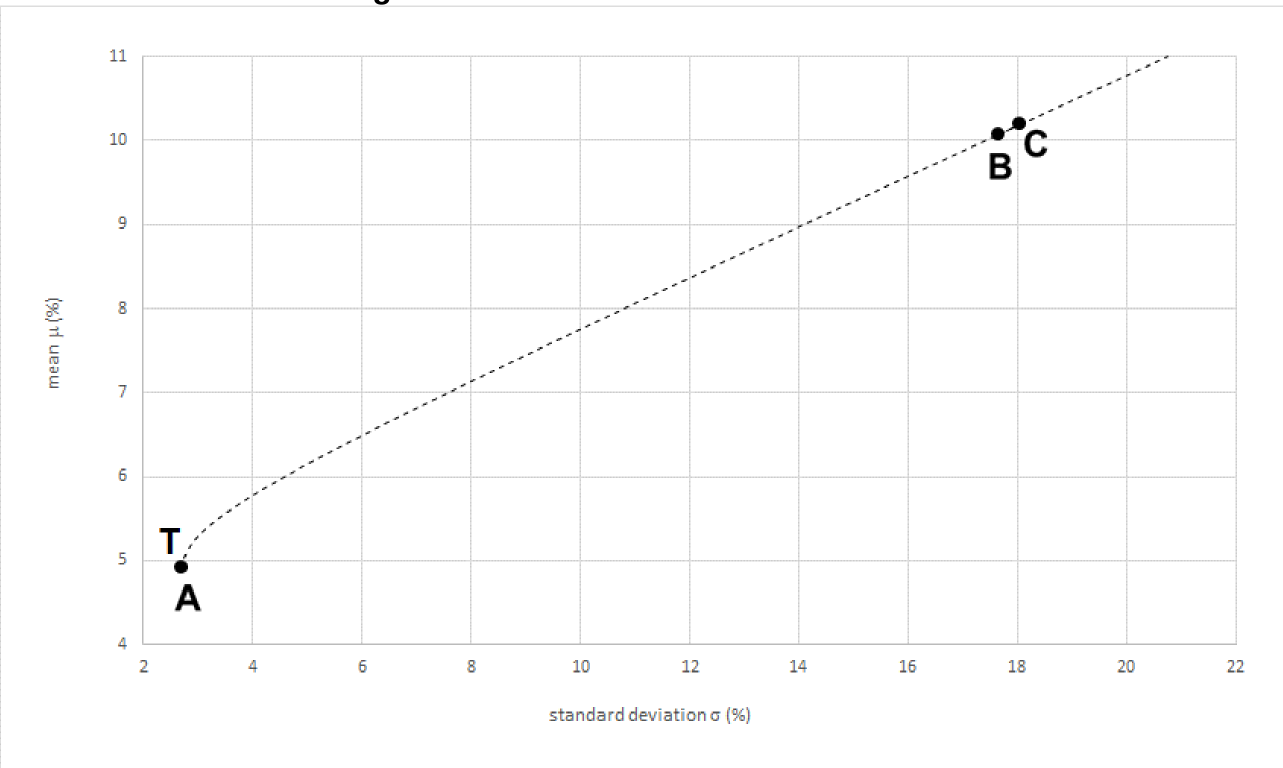
Now we can proceed as in Ghezzi (2018). At first we determine a usual efficient frontier in the space of linear moments $(\sigma; \mu)$; then we take advantage of the one-to-one mapping (5)-(6), eventually obtaining a complementary efficient frontier in the space of logarithmic moments $(s; m)$.

Restricted optimization can be carried out by applying the critical line algorithm, outlined in Markowitz (1991) and coded in Visual Basic for Applications in Markowitz and Todd (2000). However, our data sets include either 3 or 4 asset classes so that numerical optimization is also viable. Microsoft Excel Solver is helpful in this respect. Since the usual efficient frontier is piecewise hyperbolic, corner portfolios can be determined by using equations (1)-(3) appropriately.

Two usual efficient frontiers are portrayed in the $(\sigma;\mu)$ space of Figure 1, which is based on Table 1a, i.e. on the former data set. The unrestricted efficient frontier is represented by a dotted line. The restricted efficient frontier includes thehyperbolic piecesAB and BC, which have the corner portfolios A, B, and C as bounds. A is the minimum-variance portfolio, whereas C is the S&P Composite, i.e. the asset class with the highest mean.All 3 asset classes (T-bills, T-bonds, and S&P Composite) are dominant on the hyperbolic piece AB, which belongs to both efficient frontiers.Only T-bonds and S&P Composite are dominant on the hyperbolic piece BC, which is represented by a solid line. Although the corner portfolio B belongs to both efficient frontiers, the remainder of the solid line BC lies slightly below the dotted line. Therefore, each restricted efficient portfolio has a lower mean μ than the unrestricted efficient portfolio with the same standard deviation σ. The latter portfolio requires a modest short position in T-bills along with a long position in T-bonds and S&P Composite.

Remarkably, A is both the unrestricted and restricted minimum-variance portfolio. Moreover, the latter data set is such that the two portfolios are the same. In contrast, Markowitz (1991) provides a numerical example where the two minimum-variance portfolios are different, i.e. where the unrestricted minimum-variance portfolio has both positive and negative weights. Actually, if two asset classes are highly correlated and short positions are not ruled out, unrestricted optimization may call for near arbitrage, namely for going long the more rewarding asset class and short the less rewarding one. However, near arbitrage needs to take place in the very short term, whereas a strategic asset allocation is concerned with the very long term.

Figure 1: Usual Efficient Frontiers



Some corner portfolios are compared in Table 3 as well. The efficient portfolio T maps into the minimum-variance portfolio in the $(s;m)$ space, which is portrayed in Figure 2; it has been determined by following the procedure stated in Ghezzi (2018).

Notice that portfolio T belongs to both efficient frontiers of Figure 1; portfolios A and T are slightly differentso that only A is the minimum-variance portfolio in the space of linear moments.

The two usual efficient frontiers of Figure 1 map into the two complementary efficient frontiers of Figure 2. More precisely, the dotted line, i.e. the unrestricted efficient frontier, maps into the dotted line. The hyperbolic pieces TB and BC of the restricted efficient frontier map into the pieces TB and BC, which have the corner portfolios T, B, and C as bounds. Although the corner portfolio B of Figure 2 belongs to both efficient frontiers, the remainder of the solid line BC lies slightly below the dotted line.

Remark 1: Figure 2 might be misleading. The unrestricted efficient frontier is not upward sloping since it attains a local maximum $m = 9.97\%$ at $s = 37.15\%$, which is not displayed. The corresponding asset allocation is

$$w_1 = -145.27\%;\ w_2 = -4\%;\ w_3 = 249.26\%$$

i.e. it includes a large short position in T-bills and a short position in T-bonds.

Remark 2: The weights w_1, w_2, w_3 of Table 3 are determined once and for all when deriving the usual efficient frontier in the space of linear moments $(\sigma; \mu)$. The logarithmic moments m and s of Table 3 are subsequently obtained by using equations (5)-(6).

Figure 2: Complementary Efficient Frontiers

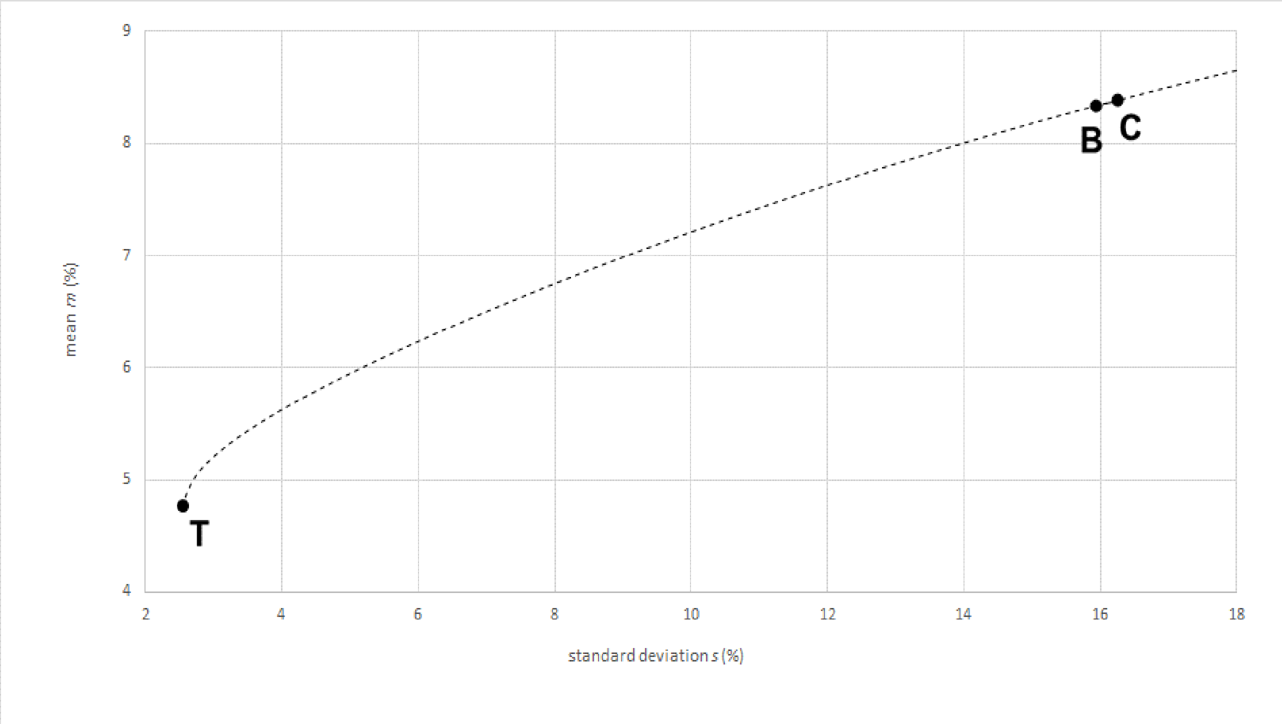


Table 3: Some Corner Portfolios

	w_1	w_2	w_3	μ	σ	m	s
A	90.55%	6.00%	3.46%	4.906%	2.695%	4.756%	2.569%
T	90.44%	5.99%	3.57%	4.912%	2.695%	4.762%	2.568%
B	0%	2.16%	97.84%	10.08%	17.66%	8.33%	15.94%
C	0%	0%	100%	10.19%	18.04%	8.38%	16.26%

The mapping between Figure 1 and Figure 2 obeys the following propositions.

Proposition 1: All efficient portfolios between points A and T map into minimum-variance portfolios in the space of logarithmic moments $(s; m)$.

Proof: As recalled in Section 2, the minimum-variance set is piecewise hyperbolic in the space of linear moments $(\sigma; \mu)$. Each hyperbolic piece maps into minimum-variance portfolios in $(s; m)$ space owing to Proposition 1 (Ghezzi, 2018), restated in Section 2. □

Proposition 2: If each hyperbolic piece of arestricted efficient frontier is such that

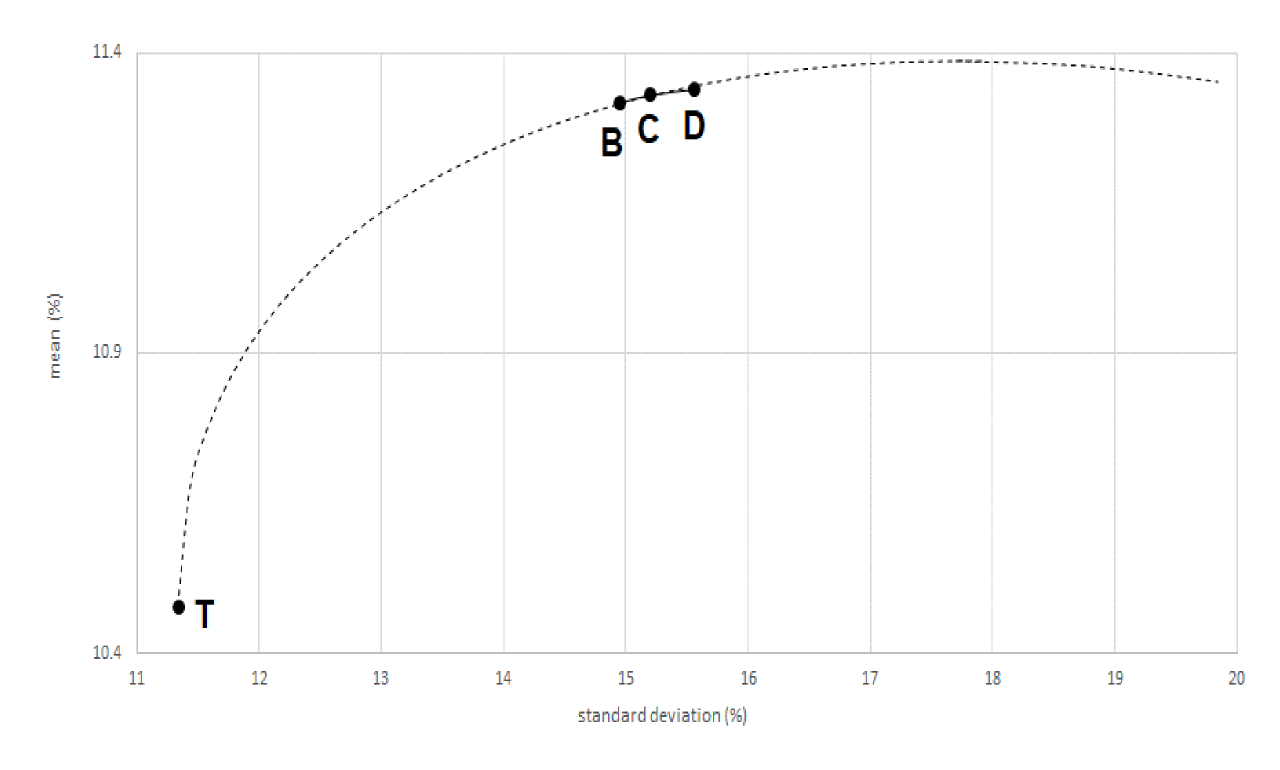
- either the discriminant (4) is negative,
- or the discriminant (4) is positive, yet the hyperbolic piece would attain a local maximum beyond the corner portfolio on the right,

all efficient portfolios above point *T* map into efficient portfolios in the space of logarithmic moments $(s; m)$.

Proof: Each hyperbolic piece in $(\sigma; \mu)$ space maps into efficient portfolios in $(s; m)$ space owing to Proposition 2 and 3 (Ghezzi 2018), restated in Section 2. □

Propositions 1 and 2 can also be applied to the latter data set, which includes four major equity classes. Starting from Table 1b two other complementary efficient frontiers can be obtained, as portrayed in Figure 3. The unrestricted efficient frontier is represented by a dotted line. The restricted efficient frontier is made up of 3 pieces, namely TB, BC, and CD. All 4 asset classes (US stocks, international stocks, real estate securities, and commodity-linked securities) are dominant on the piece TB, which belongs to both efficient frontiers. Only US stocks, international stocks, and real estate securities are dominant on the piece BC, which is represented by a solid line. Only international stocks and real estate securities are dominant on the piece CD, which is also represented by a solid line.

Figure 3: Other Complementary Efficient Frontiers



Remark 3: The unrestricted efficient frontier of Figure 3 attains a local maximum $m = 11.39\%$ at $s = 17.70\%$. However, the asset allocation is

$$w_1 = -10.40\%;\ w_2 = 6.81\%;\ w_3 = 114.58\%;\ w_4 = -10.99\%$$

i.e. it includes short positions in US stocks and commodity-linked securities.

Since Proposition 2 is met by both data sets, the complementary efficient frontiers of Figures 2 and 3 are upward sloping. Since Proposition 3 in Ghezzi (2018) is also met by both data sets, the unrestricted efficient frontiers of Figures 2 and 3 are not upward sloping.This is the main difference from previous theoretical findings.

5. Additional Empirical Findings

The theoretical findings of Section 4 rest on the assumption that portfolio returns are lognormally distributed. Therefore, we want to test the null hypothesis whereby the logarithmic returns of efficient portfolios are normally distributed.Accordingly, a few efficient portfolios are selected in both data sets. The following statistical tests are performed by using Rstudio software

- Jarque-Bera,
- Lilliefors,
- Shapiro-Wilk,
- Cramer-von Mises.

For instance, Jarque-Bera test is presented in Brooks (2014), Lilliefors test is examined in Dudewicz and Mishra (1988), Shapiro-Wilk test is presented in Bickel and Doksum (1977). Lilliefors test extends Kolmogorov-Smirnov test, also examined in Dudewicz and Mishra (1988). If each statistical test results in a p-value that is greater than 0.05, the null hypothesis is accepted.We have chosenseven efficient portfolios that have operationally importantarithmetic means; four(three) of them come from the former (latter) data set. The p-values of 7sign-restricted and efficient portfolios are reported in Table 4.

Table 4(a): p-values
Three Major US Asset Classes, Sign-Restricted and Efficient Portfolios

w_1	w_2	w_3	μ	σ	Jarque Bera	Lilliefors	Shapiro Wilk	Cramer von Mises
71.38%	5.19%	23.43%	6%	4.57%	0.1707	0.4869	0.2428	0.5331
53.87%	4.44%	41.68%	7%	7.56%	0.7136	0.9719	0.15	0.8688
36.37%	3.70%	59.93%	8%	10.79%	0.0284	0.5093	0.1092	0.5936
18.86%	2.96%	78.18%	9%	14.08%	0.0062	0.5942	0.0514	0.3909

Table 4(b): p-values
Four Major Equity Classes, Sign-Restricted and Efficient Portfolios

w_1	w_2	w_3	w_4	μ	σ	Jarque Bera	Lilliefors	Shapiro Wilk	Cramer von Mises
32.22%	2.16%	42.57%	23.05%	12.00%	12.85%	0.0000	0.0149	0.0000	0.0014
8.94%	4.70%	81.90%	4.46%	13.00%	15.87%	0.0000	0.0395	0.0004	0.0047
2.77%	5.22%	92.01%	0%	13.25%	17.08%	0.0000	0.0392	0.0009	0.0041

More precisely, Table 4a is based on a larger sample, including 140 annual returns, whereas Table 4b is based on a smaller sample, including 46 annual returns. Going down each table is like moving rightward on an efficient frontier. Notice that the efficient portfolios of Table 4a are different from the corner portfolios of Table 3.

According to Table 4a, the null hypothesis is accepted for the first two asset allocations, whereas it is rejected for the last two asset allocations. Although the last two asset allocation displays a poor p-value in Jarque Bera test, Table 5 corroborates the evidence in favour of the operational usefulness of the lognormal mapping (5)-(6). According to Table 4b, the null hypothesis is rejected for all three asset allocations.

As recalled in Section 2, the one-to-one mapping (5)-(6) involves population moments. Accordingly, we have assumed in Section 3 that the population moments of our asset classes are provided by either Table 1a or 1b, both concerning annual linear returns. In the light of Remark 2, all linear and logarithmic moments of Table 3 are population moments. More generally, all linear and logarithmic moments of Propositions 1 and 2 in this paper are population moments.

Table 5 focuses on annual logarithmic returns and compares the estimated and historical moments of the seven sign-restricted and efficient asset allocations of Table 4.

Table 5(a): Estimated and Historical Moments of Logarithmic Returns Three Major US Asset Classes, Sign-Restricted and Efficient Portfolios

w_1	w_2	w_3	m	s	$\widehat{m}$	$\hat{s}$
71.38%	5.19%	23.43%	5.73%	4.31%	5.73%	4.34%
53.87%	4.44%	41.68%	6.52%	7.06%	6.51%	7.16%
36.37%	3.70%	59.93%	7.20%	9.96%	7.19%	10.19%
18.86%	2.96%	78.18%	7.79%	12.86%	7.76%	13.31%

For instance, m is an estimated logarithmic mean, whereas $\widehat{m}$ is a historical logarithmic mean. All differences $m - \widehat{m}$ are nonnegative, whereas all differences $s - \hat{s}$ are negative.

As for all four asset allocations of Table 5a, all differences $m - \widehat{m}$ are very small and all differences $s - \hat{s}$ are small.

Table 5(b): Estimated and Historical Moments of Logarithmic Returns Four Major Equity Classes, Sign-Restricted and Efficient Portfolios

w_1	w_2	w_3	w_4	m	s	$\widehat{m}$	$\hat{s}$
32.22%	2.16%	42.57%	23.05%	10.68%	11.44%	10.57%	13.05%
8.94%	4.70%	81.90%	4.46%	11.25%	13.97%	11.13%	15.26%
2.77%	5.22%	92.01%	0%	11.32%	14.99%	11.18%	16.61%

Altogether, the null hypothesisis

- accepted for the first two asset allocations of Tables 4a and 5a. Each asset allocation displays 3 positive weights, i.e. it takes advantage of the risk-mitigating effects of a full diversification;
- rejected for the last two asset allocations of Tables 4a and 5a owing to mixed statistical evidence. Indeed, the lognormal mapping (5)-(6) seems to be operationally usefulin the light of the small differences $m - \widehat{m}$ and $s - \hat{s}$ of Table 5a;
- rejected for all major US asset classes, i.e. for all constituent asset classes.As for T-bills and T-bonds, all p-values are much smaller than 0.05 in each test.

Figure 4(a): Quantile-Quantile Plot of the First Efficient Portfolio of Table 4a

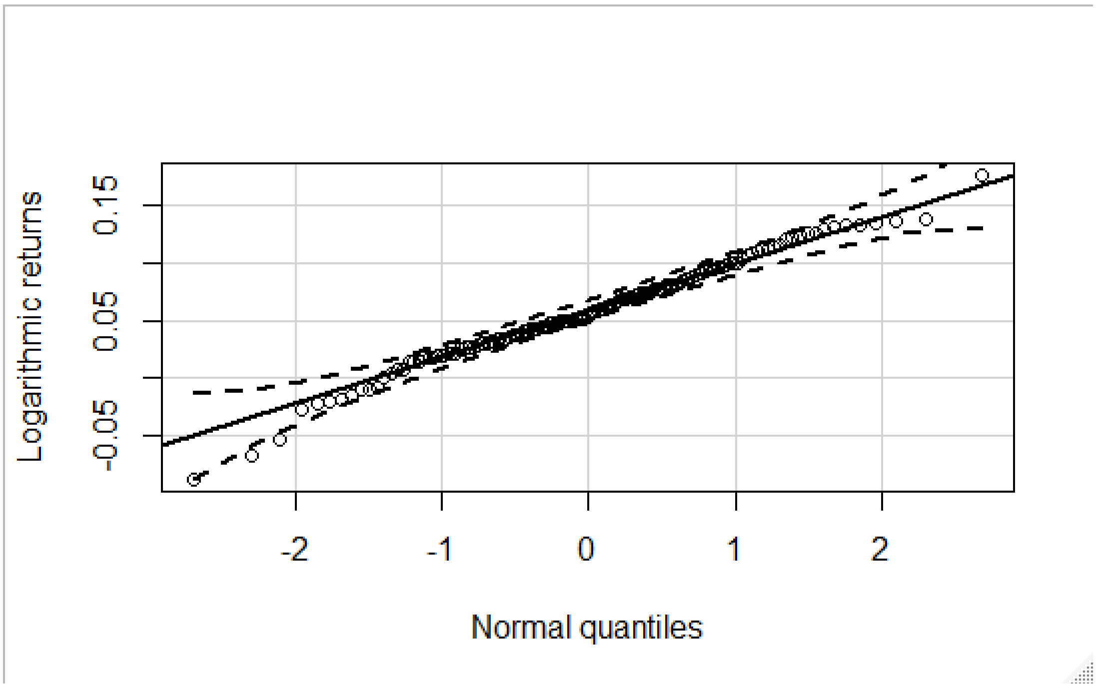
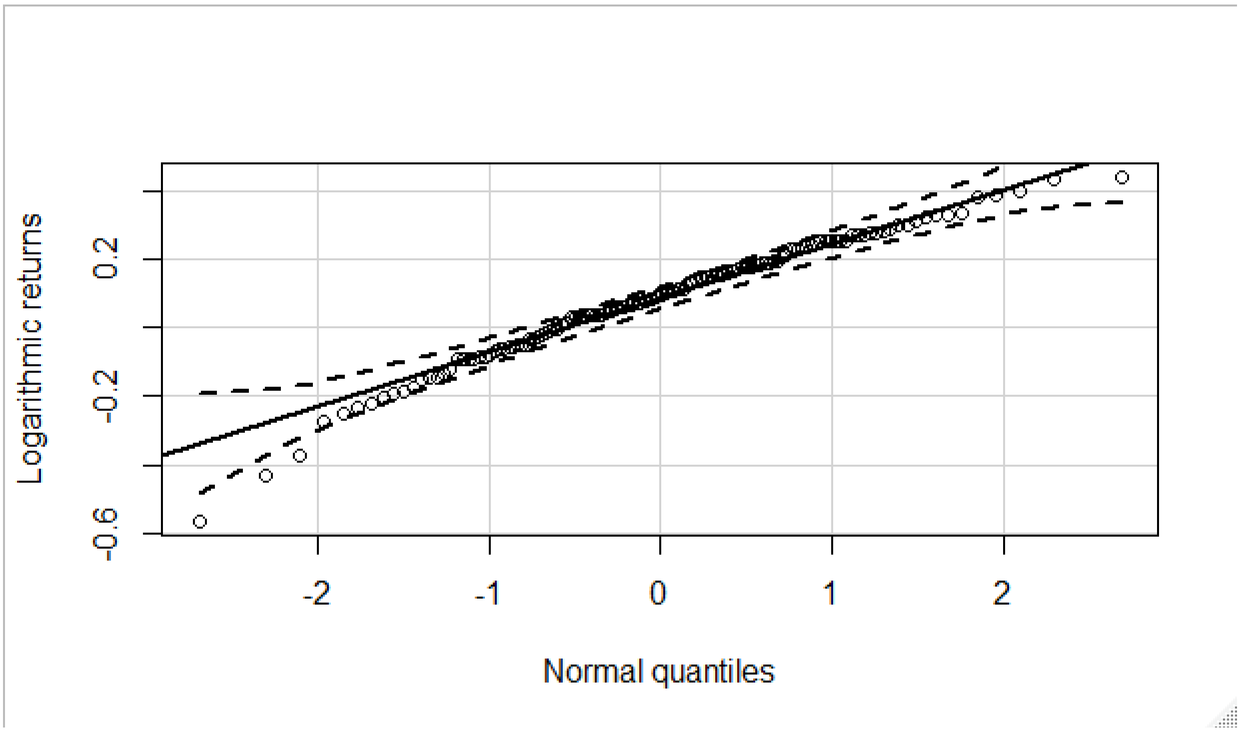


Figure 4(b): Quantile-Quantile Plot of the S&P Composite



Accordingly, two quantile-quantile plots are shown in Figure 4, where an empirical distribution of logarithmic returns is compared with a normal distribution. Figure 4a is about the first asset allocation of Tables 4a and 5a, whereas Figure 4b concerns the S&P Composite. The confidence level of the confidence band is 0.95 for both quantile-quantile plots.

6. Conclusion

Attention has been paid to a diversified portfolio of asset classes, portfolio returns being time uncorrelated and lognormal. If time is sampled once a year, weights are constant and nonnegative owing toannual rebalancing.A novel sufficient condition has been obtained. If it is met, a usual efficient frontier based on linear returns and the complementary efficient frontier based on logarithmic returns share the following properties

- they are upward sloping and made up of different pieces, each piece having two corner portfolios as bounds;
- they stretch from a corner portfolio with global minimum-variance to a corner portfolio with globalmaximum-variance;
- the two corner portfolios with global minimum-variance are different, whereas the two corner portfolios with global maximum-variance are the same, namely the asset class with the highest mean;
- if the two corner portfolios with global minimum-variance are not very different, all other corner portfolios do coincide.

In contrast, if negative weights are allowed, the complementary efficient frontier based on logarithmic returns may not be upward sloping, as proved by Ghezzi (2018). If our sufficient condition is met, the asset class with the highest mean is a corner portfolio also in the space of logarithmic moments. However, it is still unclear whether there could be stricter yet objective limits to risk taking when the sufficient condition is not fulfilled. In other words, it is still unclear whether the asset class with the highest mean could become inefficient in the space of logarithmic moments.

In our opinion, the above-mentioned findings may be useful

- in theory, as a term of comparison, since normal distributions are important in statistics;
- in practice. If the assumption of lognormal portfolio returns is tenable, a population mean of logarithmic returns is provided by our sufficient condition. Since the long-term portfolio value also depends on the sample mean of its logarithmic returns, a population mean is the best possible forecast of such a sample mean. Our sufficient condition can also shed light on the main output of Monte carlo simulation, namely a range of portfolio values and their likelihood at a set time horizon.Monte Carlo simulation allows a decision maker to take intermediate investments and divestments as well as taxes into consideration (Sharpe et al. 2007).

Needless to say, the main limitation of the mean-variance approach hasn't been overcome:the means, variances, and covariances of the linear returns of all asset classes remain hard to estimate.Altogether, the assumption of lognormal portfolio returns seems to deserve further empirical examination. According to our statistical tests, such an assumption seems more likely to be accepted or be operationally useful, whenever a large sample and appropriate diversification across asset classes are taken into consideration. In other words, the diversification in time as well as across asset classes seems to matter; remarkably, the risk-mitigating effects of a broad diversification are well known (Gibson 2008).

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