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Investment Choices, and Risk-adjusted Performance Measures with Skewness

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Asset returns are involved with multi-dimensional risks and therefore any good risk-adjusted performance measure (RAPM) should incorporate these risks. We extend the risk-adjusted performance measure theory developed by Xiang, Liu and Wang (2012) to include multiple risks. The following models are derived in this paper: a RAPM with skewness under individual equilibrium for an individual investor, a RAPM that comprises the skewness under market equilibrium for all investors using a three-moment asset pricing model. Both the individual and market RAPMs show that investors prefer a positive skewness to a negative skewness, ceteris paribus. We prove that the RAPM is the Sharpe ratio when the return doesn't show any skewness. Finally this paper provides a theoretical justification to the relationships between investment choices, an asset pricing model, an individual RAPM, and a market RAPM.

Key words: risk-adjusted investment performances, Sharpe ratio with skewness, and theoretical derivation of investment performance measures.

1. Introduction

A risk-adjusted performance measure (RAPM) is a statistic to summarize a portfolio's performance and risks and good example is the Sharpe ratio (Sharpe 1966). The RAPM can be calculated for one unit of the risks so that portfolio performances with different sizes of risks can be compared against. The Sharpe ratio is probably the most widely used RAPM in the investment management industry.

A garden variety of RAPMs have been developed based on some economic rationale or specific utility functions: i.e., the Sharpe ratio, Treynor Ratio (Treynor 1965), and Jensen Alpha (Jensen 1968), the conditional Sharpe ratio (Gregoriou and Gueyie 2003), Omega (Shadwick and Keating 2002), Lambda (Kaplan 2005), a modified Sortino ratio (Pedersen and Satchell 2002), and others (Kaplan and Knowles 2004; Young 1991; Burke 1994; Kestner 1996; Dowd 2000; Agarwal and Naik 2004; Sortino and Price 1994; and Sharma, 2004). Unfortunately there is no consensus on which RAPM provides the best measure of risk-adjusted returns. This paper strives to fill in the gap of the existing literature.

There are a number of reasons why there has been not much consensus on which RAPM being the best representation of true risk-adjusted portfolio performance. First, most of RAPMs don't have links to investors' investment choices. Since investment choice is an essential part of any investment decision process, RAPMs that do not

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factor the investment choice is not sufficient. Second, investors have different liabilities. Conservative investment shops such as insurance companies seek investments with high returns after matching the liabilities with their assets. They need more sophisticated investment strategies to accomplish this goal and the existing RAPMs lack measure the performance of their strategies. Third, with the fast growing of alternative investments in recent years, investors have more investment products to choose from. These products often show some degree of skewness and kurtosis in their return distributions. The classical RAPMs are no longer able to handle this non-normality in the distributions. Fourth, the majority of the existing RAPMs have only one risk component while investors in reality always face a number of risks in their investments. A RAPM can only inaccurately assess an investment performance if it cannot handle more than one risk. Finally, investors are inclined to use a familiar RAPM they have been using for a long time such as the Sharpe ratio despite the measure's limitations. Investors frequently hesitate to accept a RAPM if they don't understand the measure, or there is a sound theory to support the measure.

Given the above-mentioned limitations of existing literature, this paper aims to develop new RAPMs that link to investors' investment choices and incorporate multiple risks. Xiang, Liu and Wang (2012) develop a RAPM theory in a representative agent economy. They use an expected utility to model an investment choice without identifying a form of the utility function. They assume that an investor (or agent) knows his return and risk measures, funding type, and investment product (or financial market) to invest. The investor's objective is to maximize his expected utility for a given funding type. With this theory, Xiang, Liu and Wang (2012) derive several equations a RAPM must satisfy. At the same time, they examine whether a solution to the equations is the investor's RAPM. They also use the theory to derive the RAPM Omega and the more general Kappa by applying the variational principle to the utility function with respect to the investment choice, which is comprised of the process of investment decision making which consists of how the investment is funded and the risk and return attributes and has less restrictive assumptions. One flip side of their theory is that they deal with only one risk measure, therefore it is necessary to extend the theory to derive new RAPMs with multiple risks.

For the reason that unconditional return distributions of financial assets are not normal[i], most of RAPMs are insufficient to measure and evaluate an investment performance. This motivates us to incorporate a return skewness into the Sharpe ratio. We are able to derive a RAPM that includes the first three central moments of a return distribution. The three-moment RAPM has a parameter to gauge an investor's skewness preference. In order to understand the skewness preference, we develop the RAPM under a market equilibrium condition. The fact that the RAPM has three moments leads us to use a three-moment asset pricing model to infer the preference. Harvey and Siddique (2000) present a model among many three-moment asset pricing models[ii]. They derive the model by assuming the pricing kernel is a quadratic function in the return of a market portfolio. We choose their model for our purpose because the model has an explicit expression and is verified with other theory[iii].

The rest of the paper is organized into four sections. In the second section, we extend the RAPM theory to include multiple risk measures. We use an example to show how to use the extended theory to derive new RAPMs. We also demonstrate that the extended theory is different from the original one in which a RAPM must be a function of another RAPM under an investment choice. In the third section, we use the extended theory to infer a three-moment RAPM under individual equilibrium condition. In the fourth section, we derive the three-moment RAPM under market equilibrium using Harvey and Siddique's asset pricing model. We also check the robustness of the RAPM. We conclude in the last section.

2. The Extended RAPM Theory

In this section, we extend the RAPM theory by Xiang, Liu and Wang (2012) to include more than one risk measure. We derive two equations a RAPM must satisfy in the theory. We find all solutions to the equations by assuming that the return and risk measures satisfy homogenous conditions. To demonstrate the theory, we use an example to show how to develop new RAPMs. We also prove that one RAPM may not be a function of another RAPM under an investment choice in the extended theory.

We first need some notations and assumptions to model an investment choice. Let a financial market or product be an investment set consisting of $n+1$ assets which are eligible for the investor to freely trade[iv]. The investor desires to invest a portfolio in the investment set and holds the portfolio over a discrete period. The investor expects that the ith asset has a return $E(r_i)$ over the period and the portfolio has the asset at the weight w_i in the beginning of the period, $i = 0,1,2,\cdots,n$. The investor can be funded or unfunded according to the positions in the portfolio being 100% net long or not. In other words, the investor is funded when the weight constraint is $\sum_{i=0}^{n} w_i = 1$ and unfunded or dollar neutral when the weight constraint is $\sum_{i=0}^{n} w_i = 0$.

We also make the following assumptions to derive the equations which the investor's RAPM must satisfy.

1) The investor cares only about the return and risks of the investment. It means that the utility function for the investment has variables: return and risks[v].
2) The portfolio has only one constraint: the weight constraint.
3) The investor maximizes expected utility given the investment set and weight constraints

There are two investment problems in a representative agent economy. The problem for a funded investor is

$$\text{Maximize: } E(U(r_p)) = f(u, v_1, \cdots, v_m)$$

$$\text{Subject to } \begin{cases} u = \varphi(w_0, w_1, \cdots, w_n) \\ v_j = \psi_j(w_0, w_1, \cdots, w_n), j = 1, \cdots m \\ \sum_{i=0}^{n} w_i = 1 \end{cases}$$

Where f is the investor's utility function[vi], and u and $v_j, j=1,\cdots,m$ are corresponding to the return and risk variables which are the functions of the weights $w_i, i=0,1,2,\cdots,n$. The shape of the utility function represents the investor's risk aversion towards to the investment and the market. Since the function is not identified, the risk aversion is unknown.

A Lagrangian for the given maximization problem is

$$L = f\left(\varphi(w_0,\cdots,w_n),\psi_1(w_0,\cdots,w_n),\cdots,\psi_m(w_0,\cdots,w_n)\right) - \lambda\left(\sum_{i=0}^{n} w_i - 1\right),$$

Where λ is the Lagrange multiplier. From the first order conditions for the Lagrangian, we obtain the investor's equilibrium condition:

$$\begin{cases} \dfrac{\partial L}{\partial w_i} = \dfrac{\partial f}{\partial u}\dfrac{\partial \varphi}{\partial w_i} + \sum_{j=1}^{m} \dfrac{\partial f}{\partial v_j}\dfrac{\partial \psi_j}{\partial w_i} - \lambda = 0, i = 0,1,\cdots,n \\ \dfrac{\partial L}{\partial \lambda} = \sum_{i=0}^{n} w_i - 1 = 0 \end{cases} \qquad (2.1)$$

After subtracting the first equation from the first $n+1$ equations in (2.1), we have

$$\left(\frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial f}{\partial u} + \sum_{j=1}^{m}\left(\frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial f}{\partial v_j} = 0, i = 0,1,\cdots,n \qquad (2.2)$$

Multiplying the ith equation in (2.2) by the weight w_i, and then summing all the resulting equations, we can come to the following equation by the weight constraint $\sum_{i=0}^{n} w_i - 1 = 0$.

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial f}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial f}{\partial v_j} = 0 \qquad (2.3)$$

Additionally, if the investor has the function $z = z(u, v_1 \cdots, v_m)$ as his RAPM for the investment, the value of the utility function should increase most as the value of the RAPM increases[vii]. Consequently, the gradient of the function z must have the same direction as the gradient of the utility function f. Since the condition in (2.3) means that the gradient of f.

$$\nabla f = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v_1}, \cdots, \frac{\partial f}{\partial v_m}\right)^T$$

And the vector

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}, \sum_{i=0}^{n} w_i \frac{\partial \psi_1}{\partial w_i} - \frac{\partial \psi_1}{\partial w_0}, \cdots, \sum_{i=0}^{n} w_i \frac{\partial \psi_m}{\partial w_i} - \frac{\partial \psi_m}{\partial w_0}\right)^T$$

Are orthogonal, where the symbol ^{T} is the transpose operator of a vector, the gradient of function z

$$\nabla z = \left(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v_1}, \cdots, \frac{\partial z}{\partial v_m}\right)^T$$

Must Be Orthogonal To The Vector. So We Have:

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial z}{\partial v_j} = 0 \qquad (2.4)$$

Which is the differential equation the RAPM must satisfy for the funded investor.

If the investor is unfunded, the investment problem is

Maximize: $E(U(r_p)) = f(u, v_1, \cdots, v_m)$

Subject to $\begin{cases} u = \varphi(w_0, w_1, \cdots, w_n) \\ v_j = \psi_j(w_0, w_1, \cdots, w_n), j = 1, \cdots m \\ \sum_{i=0}^{n} w_i = 0 \end{cases}$

And the Lagrangian for the given maximization problem is

$$L = f(\varphi(w_0, \cdots, w_n), \psi_1(w_0, \cdots, w_n), \cdots, \psi_m(w_0, \cdots, w_n)) - \lambda\left(\sum_{i=0}^{n} w_i\right).$$

The first order conditions for the Lagrangian is

$$\begin{cases} \dfrac{\partial L}{\partial w_i} = \dfrac{\partial f}{\partial u}\dfrac{\partial \varphi}{\partial w_i} + \sum_{j=1}^{m} \dfrac{\partial f}{\partial v_j}\dfrac{\partial \psi_j}{\partial w_i} - \lambda = 0, i = 0, 1, \cdots, n \\ \dfrac{\partial L}{\partial \lambda} = \sum_{i=0}^{n} w_i = 0 \end{cases}$$

We obtain an equation for the unfunded investor in the similar way as the case for the funded investor. The equation is

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i}\right)\frac{\partial z}{\partial v_j} = 0 \qquad (2.5)$$

The asset r_0 plays a very important role when we derive the equation in (2.4). In practice, it is often the benchmark that the investor uses in the investment. We would like to call the asset as a reference asset in the rest of this paper. Note that the reference asset must be in the investment set.

We next simplify the equations in (2.4) and (2.5) by adding some conditions so that we can find all solutions to the equations. The conditions are

$$\begin{cases} \sum_{i=0}^{n} w_i \dfrac{\partial \varphi}{\partial w_i} = \varphi \\ \sum_{i=0}^{n} w_i \dfrac{\partial \psi_j}{\partial w_i} = \psi_j \end{cases} \tag{2.6}$$

These conditions reveal an important property of the return and risk measures. We will discuss the property later on. Under the conditions in (2.6), the equation in (2.4) turns out to be

$$\left(u - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(v_j - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial z}{\partial v_j} = 0 \tag{2.7}$$

And the equation in (2.5) becomes

$$u\frac{\partial z}{\partial u} + \sum_{j=1}^{m} v_j \frac{\partial z}{\partial v_j} = 0 \tag{2.8}$$

Furthermore, if both partial derivatives $\frac{\partial \varphi}{\partial w_0}$ and $\frac{\partial \psi_j}{\partial w_0}$ are constant, $j = 1,\cdots,m$, i.e.,

$$\begin{cases} \dfrac{\partial \varphi}{\partial w_0} = a \\ \dfrac{\partial \psi_j}{\partial w_0} = b_j,\ j = 1,\cdots,m \end{cases} \tag{2.9}$$

The equation in (2.7) becomes

$$(u - a)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}(v_j - b_j)\frac{\partial z}{\partial v_j} = 0 \tag{2.10}$$

According to the Euler's Lemma, the conditions in (2.6) are equivalent to the fact that the return variable φ and risk variable $\psi_j, j = 1,\cdots,m$ are homogeneous functions of degree one. The homogeneity means

$$\varphi(c_0 w_0,\cdots,c_0 w_n)=c_0\varphi(w_0,\cdots,w_n) \qquad (2.11)$$

And

$$\psi_j(c_1 w_0,\cdots,c_1 w_n)=c_1\psi_j(w_0,\cdots,w_n),\ j=1,\cdots,m \qquad (2.12)$$

Where c_0 and c_1 are positive constants. The conditions in (2.11) and (2.12) tell that the return and risk in a portfolio should increase or decrease as the same rate as the weights do. The condition in (2.12) is consistent with the positive homogeneity axiom of a coherent risk measure defined by Artzner et. al (1999).

The conditions in (2.9) indicate that the changes in the return u and the risk variables v_j have the constant rates a and $b_j, j=1,\cdots,m$, separately, as the weight w_0 changes. The equation in (2.10) is the one in (2.8) when the constants a and $b_j, j=1,\cdots,m$, are zero. However, the two equations are for investors with different funding types.

Since the equation in (2.10) is a Lagrange's equation, it is easy to get the general solution by finding a solution to the equations

$$\frac{du}{u-a}=\frac{dv_1}{v_1-b_1}=\cdots=\frac{dv_m}{v_m-b_m}=\frac{dz}{0} \qquad (2.13)$$

The equations in (2.13) have the solution

$$\begin{cases}\dfrac{u-a}{v_1-b_1}=C_1\\ \dfrac{v_j-b_j}{v_1-b_1}=C_j,\ j=2,\cdots,m\\ z=C\end{cases}$$

Where $C_j, j=1,\cdots,m$ And C Are Constant. As A Result, The General Solution To The Equation In (2.10) Is

$$z=k\left(\frac{u-a}{v_1-b_1},\frac{v_2-b_2}{v_1-b_1},\cdots,\frac{v_m-b_m}{v_1-b_1}\right) \qquad (2.14)$$

Where k is a function. Therefore, the RAPM for the funded investor has the function form in (2.14). Similarly, we conclude that an unfunded investor has the RAPM in the form:

$$z = k\left(\frac{u}{v_1}, \frac{v_2}{v_1}, \cdots, \frac{v_m}{v_1}\right) \qquad (2.15)$$

We can use the extended theory to derive new RAPMs. For example, suppose that an investor has the expected utility

$$E(U(r_p)) = f(E(r_p), \sigma(r_p), CVaR(r_p)) \qquad (2.16)$$

Where The Portfolio Holds The Asset r_i With A Weight w_i, $i = 0,1,2,\cdots,n$ And Has The Expected Return $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$. The Risk Measures $\sigma(r_p)$ And $CVaR(r_p)$ Are Corresponding To The Volatility And The Conditional Value At Risk Of The Portfolio. In The Extended Theory, Let

$$u = \varphi(w_0, w_1, \cdots, w_n) = E(r_p),$$

$$v_1 = \psi_1(w_0, w_1, \cdots, w_n) = \sigma(r_p),$$

And

$$v_2 = \psi_2(w_0, w_1, \cdots, w_n) = CVaR(r_p).$$

Since

$$\varphi(c_0 w_0, \cdots, c_0 w_n) = E(c_0 r_p) = c_0 E(r_p) = c_0 \varphi(w_0, \cdots, w_n),$$

$$\psi_1(c_1 w_0, \cdots, c_1 w_n) = \sigma(c_1 r_p) = c_1 \sigma(r_p) = c_1 \psi_1(w_0, \cdots, w_n),$$

And

$$\psi_2(c_1 w_0, \cdots, c_1 w_n) = CVaR(c_1 r_p) = c_1 CVaR(r_p) = c_1 \psi_2(w_0, \cdots, w_n),$$

Where c_0 And c_1 Are Positive Constants, The Conditions In (2.11) And (2.12) Are Satisfied. By (2.15), If The Investor Is Unfunded And Has The Expected Utility In (2.16), The Investor's Rapm Has The Form:

$$z = k\left(\frac{E(r_p)}{\sigma(r_p)}, \frac{CVaR(r_p)}{\sigma(r_p)}\right).$$

We have many options to choose the form of the RAPM. For examples, we can select the RAPM as

$$z=\omega\frac{E(r_p)}{\sigma(r_p)}+(1-\omega)\frac{E(r_p)}{\sigma(r_p)}\Big/\frac{CVaR(r_p)}{\sigma(r_p)}=\frac{E(r_p)}{\omega\sigma(r_p)+(1-\omega)CVaR(r_p)},$$

Or

$$z=\left(\frac{E(r_p)}{\sigma(r_p)}\right)^{\omega}\left(\frac{E(r_p)}{\sigma(r_p)}\Big/\frac{CVaR(r_p)}{\sigma(r_p)}\right)^{1-\omega}=\frac{E(r_p)}{(\sigma(r_p))^{\omega}(CVaR(r_p))^{1-\omega}}$$

Where ω Is A Weight Between The Zero And One.

For a funded investor, we presume that the asset r_0 is a risk-free asset r_f. Because

$$\frac{\partial\varphi}{\partial w_0}=\frac{\partial(E(r_p))}{\partial w_0}=\frac{\partial\left(\sum_{i=0}^{n}w_iE(r_i)\right)}{\partial w_0}=E(r_0)=r_f,$$

$$\frac{\partial\psi_1}{\partial w_0}=\frac{\partial\sigma(r_p)}{\partial w_0}=\frac{\partial\sigma\left(\sum_{i=0}^{n}w_ir_i\right)}{\partial w_0}=\frac{\partial\sigma\left(\sum_{i=1}^{n}w_ir_i\right)}{\partial w_0}=0,$$

And

$$\frac{\partial\psi_2}{\partial w_0}=\frac{\partial CVaR(r_p)}{\partial w_0}=\frac{\partial CVaR\left(\sum_{i=0}^{n}w_ir_i\right)}{\partial w_0}=\frac{\partial CVaR\left(\sum_{i=1}^{n}w_ir_i\right)}{\partial w_0}=0,$$

The conditions in (2.9) hold. By (2.14), the funded investor has the RAPM with the form:

$$z=k\left(\frac{E(r_p)-r_f}{\sigma(r_p)},\frac{CVaR(r_p)}{\sigma(r_p)}\right).$$

Similar to the unfunded case, we may choose one of the following RAPMs as the investor's RAPM:

$$z=\frac{E(r_p)-r_f}{\omega\sigma(r_p)+(1-\omega)CVaR(r_p)} \qquad (2.17)$$

And

$$z=\frac{E(r_p)-r_f}{(\sigma(r_p))^{\omega}(CVaR(r_p))^{1-\omega}} \qquad (2.18)$$

Xiang, Liu and Wang (2012) show that a RAPM is a strictly increasing function of a preferred RAPM when an investment choice includes only one risk measure. This isn't true in the extended theory. For instance, if there is a preferred RAPM in the above example, both the RAPMs in (2.17) and (2.18) should be strictly increasing functions of the preferred RAPM. Therefore, one RAPM is a function of another RAPM, that is, there is a function g such that

$$\frac{E(r_p)-r_f}{\omega\sigma(r_p)+(1-\omega)CVaR(r_p)}=g\left(\frac{E(r_p)-r_f}{(\sigma(r_p))^{\omega}(CVaR(r_p))^{1-\omega}}\right).$$

It follows that $g(x)=x$ when $\sigma(r_p)=CVaR(r_p)$. Hence, we have

$$\frac{E(r_p)-r_f}{\omega\sigma(r_p)+(1-\omega)CVaR(r_p)}=\frac{E(r_p)-r_f}{(\sigma(r_p))^{\omega}(CVaR(r_p))^{1-\omega}},$$

Which implies $\omega\sigma(r_p)+(1-\omega)CVaR(r_p)=(\sigma(r_p))^{\omega}(CVaR(r_p))^{1-\omega}$. This is untrue when $0<\omega<1$ and $\sigma(r_p)\neq CVaR(r_p)$. As a result, the extended theory doesn't guarantee that there exist a preferred RAPM. It also ensures that a RAPM may not be a function of another RAPM.

We conclude in this section that a RAPM must satisfy one of the equations in (2.8) and (2.10) if an investor has the expected utility: $E(U(r_p))=f(u,v_1,\cdots,v_m)$. The RAPM has the form in (2.14) for a funded investor under the conditions in (2.9), (2.11) and (2.12) while it has the form in (2.15) for an unfunded investor under the conditions in (2.11) and (2.12). In addition, we prove that a preferred RAPM doesn't exist if an investment choice involves more than one risk measures.

3. The Individual RAPM

This section uses the extended RAPM theory to infer a three-moment RAPM under an individual equilibrium condition. The individual RAPM has a skewness preference associated with each investor. The skewness preference measures the degree of skewness the investor can tolerate in an investment. We derive the RAPM utilizing the following three forms: the RAPM is an extension of the Sharpe ratio, i.e., it is the Sharpe ratio if the return shows a small degree of skewness; the RAPM has a simple expression; and an investor can explain the RAPM in an economic way.

Suppose that a funded investor has the expected utility

$$E(U(r_p))=f(E(r_p),\sigma(r_p),\gamma(r_p)) \tag{3.1}$$

Where $\gamma(r_p)=\left[E(r_p-E(r_p))^3\right]^{\frac{1}{3}}$ is the skewness of the portfolio return[viii], and the reference asset r_0 is a risk-free asset r_f. In the extended theory, we can assume that

$$u = \varphi(w_0, w_1, \cdots, w_n) = E(r_p),$$

$$v_1 = \psi_1(w_0, w_1, \cdots, w_n) = \sigma(r_p),$$

And

$$v_2 = \psi_2(w_0, w_1, \cdots, w_n) = \gamma(r_p).$$

From the example in the previous section, we know that $u = E(r_p)$ and $v_1 = \sigma(r_p)$ satisfy the condition in (2.11), the condition in (2.12), and conditions in (2.9), respectively. Since

$$\psi_2(c_1 w_0, \cdots, c_1 w_n) = \left[E\left(c_1 r_p - E(c_1 r_p)\right)^3\right]^{\frac{1}{3}} = c_1\left[E\left(r_p - E(r_p)\right)^3\right]^{\frac{1}{3}} = c_1 \psi_2(w_0, \cdots, w_n)$$

And

$$\frac{\partial \psi_2}{\partial w_0} = \frac{\partial \gamma(r_p)}{\partial w_0} = \frac{\partial \gamma\left(\sum_{i=0}^{n} w_i r_i\right)}{\partial w_0} = \frac{\partial \gamma\left(\sum_{i=1}^{n} w_i r_i\right)}{\partial w_0} = 0,$$

$v_2 = \gamma(r_p)$ satisfies the condition in (2.12) and the second condition in (2.9). Therefore, the funded investor has the following three-moment RAPM based on the result in (2.14).

$$z = k\left(\frac{E(r_p) - r_f}{\sigma(r_p)}, \frac{\gamma(r_p)}{\sigma(r_p)}\right) \tag{3.2}$$

Obviously, the RAPM consists of the first three central moments of return. To order to find a simple RAPM for the funded investor, we initially consider this kind of the function k, $k(x, y) = k_1(x)k_2(y)$, where k_1 and k_2 are two functions in single variable. By (3.2), the RAPM is

$$z = k_1\left(\frac{E(r_p) - r_f}{\sigma(r_p)}\right) k_2\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right).$$

In view of the fact that the RAPM is the Sharpe ratio if the portfolio returns do not show any skewness, we have

$$k_1\left(\frac{E(r_p) - r_f}{\sigma(r_p)}\right) = \frac{E(r_p) - r_f}{\sigma(r_p)}$$

And $k_2(0) = 1$.

As the volatility $\sigma(r_p)$ is the squared root of the second central moment and the skewness $\gamma(r_p)$ is the cubed root of the third central moment, we suppose that $k_2(y) = a + by + cy^2 + dy^3$ by approximating k_2 with the 3rd degree Taylor polynomial, where a, b, c and d are constants. It follows immediately from $k_2(0) = 1$ that $a = 1$ that the RAPM is

$$z = \frac{E(r_p) - r_f}{\sigma(r_p)} \left(1 + b\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right) + c\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2 + d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3 \right) \qquad (3.3)$$

Notice that the terms $(\gamma(r_p)/\sigma(r_p))^2$ and $(\gamma(r_p)/\sigma(r_p))^3$ are smaller than $\gamma(r_p)/\sigma(r_p)$ at the order one and two respectively when the skewness is closer to zero. If the investor does not care a small skew of return, but does mind a larger skew, we can assume that $b = 0$. So the RAPM in (3.3) turns into

$$z = \frac{E(r_p) - r_f}{\sigma(r_p)} \left(1 + c\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2 + d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3 \right).$$

Recall that the gradients of the RAPM and the expected utility have the same direction under the individual equilibrium condition. The RAPM maximization is the same as the expected utility maximization. Therefore, we can treat the RAPM as an expected utility. The RAPM must satisfy the expected utility properties of non-satiation, decreasing marginal utility, and non-increasing absolute risk aversion (i.e., risk aversion should not increase if wealth increases). The first two properties are equivalent to the conditions $\partial z/\partial E(r_p) > 0$ and $\partial z/\partial \sigma(r_p) < 0$, correspondingly. The third property implies the condition $\partial z/\partial \gamma(r_p) \geq 0$ [ix]. Since

$$\frac{\partial z}{\partial \gamma(r_p)} = \frac{E(r_p) - r_f}{\sigma(r_p)^2} \left(2c\frac{\gamma(r_p)}{\sigma(r_p)} + 3d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2 \right)$$

And the skew $\gamma(r_p)$ can be positive or negative, the constant c must be zero and d must be non-negative in order to $\partial z/\partial \gamma(r_p) \geq 0$. In this case, the RAPM becomes

$$z = \frac{E(r_p) - r_f}{\sigma(r_p)} \left(1 + d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3 \right) \qquad (3.4)$$

The non-negative constant d in (3.4) shows that an investor prefers the portfolio return with a positive skew to a negative skew if other things are equal. In other word, an investor demands a risk premium when a return has a negative skew. So, the constant is a gauge an investor's attitude towards to the skewness of a return. The investor can use a small constant if believing the skewness has a small impact on his portfolio while putting a larger constant if believing the skewness has a big impact on the portfolio. However, to guarantee the two conditions $\partial z/\partial E(r_p) > 0$ and $\partial z/\partial \sigma(r_p) < 0$, an investor has to select the constant satisfying $1 + 4d\left(\gamma(r_p)/\sigma(r_p)\right)^3 > 0$. Please remember that the expression $\left(\gamma(r_p)/\sigma(r_p)\right)^3$ in (3.4) is the normalized skewness by the convention in this paper (please see the footnote 8).

Similarly, we can infer a three-moment RAPM for an unfunded investor. We skip the detail to infer the RAPM here. The RAPM has the expression.

$$z = \frac{E(r_p)}{\sigma(r_p)}\left(1 + d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3\right) \qquad (3.5)$$

In this section, we started with the general form in (3.2) to determine new three-moment RAPMs under individual equilibrium condition. An investor should use the RAPM in (3.4) for a funded investment and the RAPM in (3.5) for an unfunded investment if he has the expected utility in (3.1). Also, the investor is free to choose his skewness preference in these RAPMs.

4. The Market RAPM

The approach to derive the RAPM in the previous section has a very sound theory behind it. However, the RAPM is used only by an individual investor due to its skewness preference associated with the investor. To broaden it and to get a consensus on the skewness preference among various investors collectively, we are going to derive the RAPM under a market equilibrium condition. This approach not only gives the explicit expression of the preference, but also explains the meaning of the preference such as how the preference links to market timing. Additionally, we will verify the robustness of the RAPM in terms of properties of an expected utility.

Harvey and Siddique (2000) provide an asset pricing model that incorporates conditional skewness. To derive the model, they use the first order condition $E\left((1 + r_{p,t+1})m_{t+1} \middle| \Omega_t\right) = 1$ for an investor holding a risky asset r_p and assume the pricing kernel m_{t+1} is a quadratic function in the first-order return r_b of a market portfolio. We like to use this model to infer the RAPM because the model has an explicit expression. We copy the original model without the time subscripts as follows.

$$E(r_p) - r_f = \lambda_1 Cor(r_{p,} r_b) + \lambda_2 Cov\left(r_p, (r_b - E(r_b))^2\right) \qquad (4.1)$$

Where

$$\lambda_1 = \frac{Var\left((r_b - E(r_b))^2\right)E(r_b - r_f) - Skew(r_b)E\left((r_b - E(r_b))^2 - r_f\right)}{Var(r_b)Var\left((r_b - E(r_b))^2\right) - (Skew(r_b))^2} \qquad (4.2)$$

And

$$\lambda_2 = \frac{Var(r_b)E\left((r_b - E(r_b))^2 - r_f\right) - Skew(r_b)E(r_b - r_f)}{Var(r_b)Var\left((r_b - E(r_b))^2\right) - (Skew(r_b))^2} \qquad (4.3)$$

If we denote $\rho(x, y)$ as the correlation between the variables x and y, the notations in (4.1), (4.2), and (4.3) have the meanings:

$$Var(r_b) = \sigma(r_b)^2,$$

$$Var\left((r_b - E(r_b))^2\right) = \sigma\left((r_b - E(r_b))^2\right)^2,$$

$$Skew(r_b) = \gamma(r_b)^3,$$

$$Cor(r_p, r_b) = \rho(r_p, r_b)\sigma(r_p)\sigma(r_b),$$

And

$$Cor\left(r_p, (r_b - E(r_b))^2\right) = \rho\left(r_p, (r_b - E(r_b))^2\right)\sigma(r_p)\sigma\left((r_b - E(r_b))^2\right).$$

Our aim is to divide the right-side of the model in (4.1) into two components. One component explains most of the excess return $E(r_p - r_f)$ while another component only has a small impact on the excess return. We are going to infer the RAPM from the first component. To simplify notations, let

$$R_i = (r_i - E(r_i))^2 \qquad (4.4)$$

And

$$S(r_i) = \frac{\gamma(r_i)^3}{\sigma(r_i)\sigma\left((r_i - E(r_i))^2\right)} = \frac{\gamma(r_i)^3}{\sigma(r_i)\sigma(R_i)} \qquad (4.5)$$

Where r_i and R_i are the first-order and second-order returns of an risk asset, respectively.

First, we can write the right-side of the model in (4.1) as one fraction after replacing λ_1 and λ_2 in (4.1) with the two fractions in (4.2) and (4.3), and adding the two fractions together. After substituting the original notations with our notations, the numerator of the right-side in (4.1) is

$$\left\{\begin{array}{l}\left[\sigma(R_b)^2 E(r_b-r_f)-\gamma(r_b)^3 E(R_b-r_f)\right]\rho(r_p,r_b)\sigma(r_p)\sigma(r_b)\\ +\left[\sigma(r_b)^2 E(R_b-r_f)-\gamma(r_b)^3 E(r_b-r_f)\right]\rho(r_p,R_b)\sigma(r_p)\sigma(R_b)\end{array}\right\}.$$

If we rearrange the terms according to $E(r_b-r_f)$ and $E(R_b-r_f)$ respectively in the above expression and take the term $\sigma(r_p)\sigma(r_b)^2\sigma(R_b)^2$ out of the expression, the numerator becomes

$$\begin{aligned}&\left\{\begin{array}{l}\left[\sigma(R_b)^2\rho(r_p,r_b)\sigma(r_p)\sigma(r_b)-\gamma(r_b)^3\rho(r_p,R_b)\sigma(r_p)\sigma(R_b)\right]E(r_b-r_f)\\ +\left[\sigma(r_b)^2\rho(r_p,R_b)\sigma(r_p)\sigma(R_b)-\gamma(r_b)^3\rho(r_p,r_b)\sigma(r_p)\sigma(r_b)\right]E(R_b-r_f)\end{array}\right\}\\
&=\sigma(r_p)\sigma(r_b)^2\sigma(R_b)^2\left\{\begin{array}{l}\left(\rho(r_p,r_b)-\dfrac{\gamma(r_b)^3}{\sigma(r_b)\sigma(R_b)}\rho(r_p,R_b)\right)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)\\ +\left(\rho(r_p,R_b)-\dfrac{\gamma(r_b)^3}{\sigma(r_b)\sigma(R_b)}\rho(r_p,r_b)\right)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\end{array}\right\}\\
&=\sigma(r_p)\sigma(r_b)^2\sigma(R_b)^2\left\{\begin{array}{l}\left(\rho(r_p,r_b)-S(r_b)\rho(r_p,R_b)\right)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)\\ +\left(\rho(r_p,R_b)-S(r_b)\rho(r_p,r_b)\right)E\left(\dfrac{R_p-r_f}{\sigma(R_b)}\right)\end{array}\right\}\end{aligned}$$

Where we use the notation in (4.5) in the last expression. By subtracting and adding the terms $S(r_b)\rho(r_p,r_b)$ and $S(r_b)\rho(r_p,R_b)$, separately, we can change the numerator to the following equivalent expressions:

$$\begin{aligned}&\sigma(r_p)\sigma(r_b)^2\sigma(R_b)^2\left\{\begin{array}{l}\left(\rho(r_p,r_b)-S(r_b)\rho(r_p,r_b)+S(r_b)\rho(r_p,r_b)-S(r_b)\rho(r_p,R_b)\right)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)\\ +\left(\rho(r_p,R_b)-S(r_b)\rho(r_p,R_b)+S(r_b)\rho(r_p,R_b)-S(r_b)\rho(r_p,r_b)\right)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\end{array}\right\}\\
&=\sigma(r_p)\sigma(r_b)^2\sigma(R_b)^2\left\{\begin{array}{l}(1-S(r_b))\left[\rho(r_p,r_b)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)+\rho(r_p,R_b)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\\ +S(r_b)\left[\rho(r_p,r_b)-\rho(r_p,R_b)\right]\left[E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)-E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\end{array}\right\}\qquad(4.6)\end{aligned}$$

On the other hand, if using our notations and factoring the term $\sigma(r_b)^2\sigma(R_b)^2$ out of the denominator on the right-side of the model in (4.1), the denominator turn out to be

$$\left\{\begin{array}{l}\sigma(r_b)^2\sigma(R_b)^2-\gamma(r_b)^6\\ =\sigma(r_b)^2\sigma(R_b)^2\left[1-\dfrac{\gamma(r_b)^6}{\sigma(r_b)^2\sigma(R_b)^2}\right]\\ =\sigma(r_b)^2\sigma(R_b)^2\left(1-S(r_b)^2\right)\end{array}\right\}$$

Since the right-side of the model in (4.1) is the ratio of the numerator in (4.6) to the denominator, we can convert the model to

$$E(r_p-r_f)=\frac{\sigma(r_p)}{1-S(r_b)^2}\left\{\begin{array}{l}(1-S(r_b))\left[\rho(r_p,r_b)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)+\rho(r_p,R_b)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\\ +S(r_b)\left[\rho(r_p,r_b)-\rho(r_p,R_b)\right]\left[E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)-E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\end{array}\right\}$$

So the model is equivalent to

$$E(r_p-r_f)=\left\{\begin{array}{l}\dfrac{1}{1+S(r_b)}\left[\rho(r_p,r_b)\sigma(r_p)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)+\rho(r_p,R_b)\sigma(r_p)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\\ +\dfrac{S(r_b)\sigma(r_p)\left[\rho(r_p,r_b)-\rho(r_p,R_b)\right]}{1-S(r_b)^2}\left[E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)-E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\end{array}\right\} \qquad (4.7)$$

On the right-side of the model in (4.7), the first component is

$$\begin{array}{l}\dfrac{1}{1+S(r_b)}\left[\rho(r_p,r_b)\sigma(r_p)E\left(\dfrac{r_b-r_f}{\sigma(r_b)}\right)+\rho(r_p,R_b)\sigma(r_p)E\left(\dfrac{R_b-r_f}{\sigma(R_b)}\right)\right]\\ =\dfrac{1}{1+S(r_b)}\left[\beta(r_p,r_b)E(r_b-r_f)+\beta(r_p,R_b)E(R_b-r_f)\right]\end{array}$$

Where $\beta(r_p,r_b)$ and $\beta(r_p,R_b)$ are the betas of the asset r_p corresponding to the first-order return r_b and the second-order return R_b of the market portfolio. So the first component can be treated as the sum of expected excess returns of r_b and R_b due to two traditional CAPMs or one three-moment CAPM, adjusted by the term $(1+S(r_b))$. The term $S(r_b)$ will be discussed later on. We are going to show that the component is the primary contributor to the excess return $E(r_p-r_f)$.

The second component on the right-side of the model in (4.7) is

$$\frac{S(r_b)\sigma(r_p)[\rho(r_p,r_b)-\rho(r_p,R_b)]}{1-S(r_b)^2}\left[E\left(\frac{r_b-r_f}{\sigma(r_b)}\right)-E\left(\frac{R_b-r_f}{\sigma(R_b)}\right)\right].$$

The component consists of the difference between two market prices for risk: $E(r_b - r_f)/\sigma(r_b)$ and $E(R_b - r_f)/\sigma(R_b)$. We can use the no-arbitrage principle to argue that two market prices for risk must be equal. Suppose that there are two basis portfolios: one portfolio r_1 exposures to only the risk r_b and other portfolio r_2 exposures to only the risk R_b [x]. We are able to arbitrage to sell the first portfolio to finance the purchase of the second portfolio if the market price for risk $E(r_b - r_f)/\sigma(r_b)$ is higher than the market price for risk $E(R_b - r_f)/\sigma(R_b)$. We can do the opposite trade if the first market price for risk is lower than the second market price for risk. Therefore, the second component should have a small impact on the excess return $E(r_p - r_f)$ while the first component can explain the most of the excess return. As a result, we can approximately write the model as

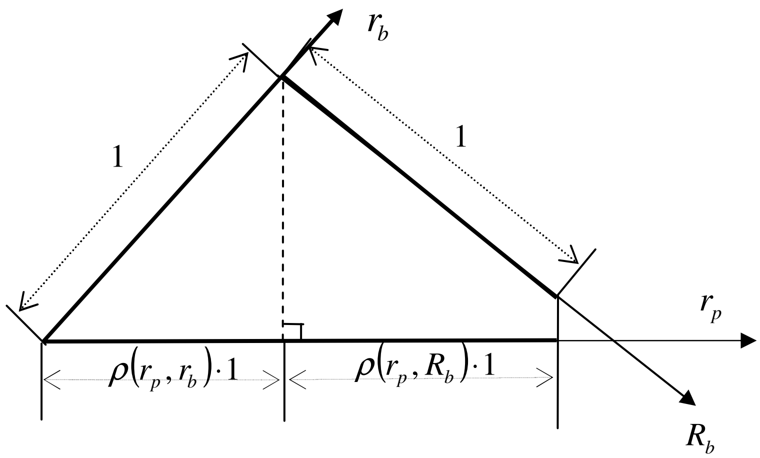
$$E(r_p - r_f) \approx \frac{1}{1+S(r_b)}[\rho(r_p,r_b)E(r_b - r_f) + \rho(r_p,R_b)E(R_b - r_f)] \qquad (4.8)$$

Which Implies

$$\frac{E(r_p - r_f)(1+S(r_b))}{\sigma(r_p)} \approx \rho(r_p,r_b)E\left(\frac{r_b-r_f}{\sigma(r_b)}\right) + \rho(r_p,R_b)E\left(\frac{R_b-r_f}{\sigma(R_b)}\right) \qquad (4.9)$$

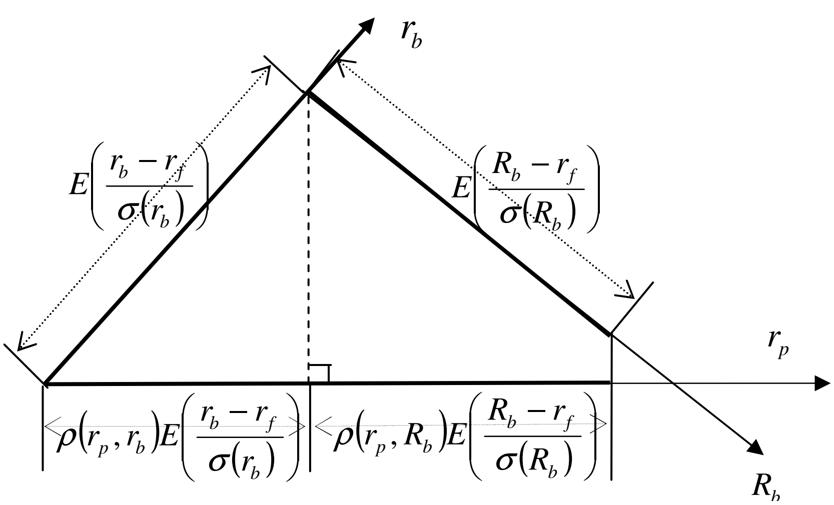
Next, we explain that the right-side in (4.9) is a risk-adjusted return of the asset r_p. Since $E((r_b - r_f)/\sigma(r_b))$ and $E((R_b - r_f)/\sigma(r_b))$ are market prices for risk, they are both risk-adjusted returns. It follows that $\sigma((r_b - r_f)/\sigma(r_b)) = 1$ and $\sigma((R_b - r_f)/\sigma(R_b)) = 1$. As the Figure 1 shows[xi], the portion in the risk $\sigma(r_p)$ explained by the unit of risk $(r_b - r_f)/\sigma(r_b)$ and the unit of risk $(R_b - r_f)/\sigma(R_b)$ is $\rho(r_p,r_b)\cdot 1 + \rho(r_p,R_b)\cdot 1$, which is the sum of two projections onto the portfolio risk r_p. The two projections are the risks projected from the two units of risks onto the portfolio risk, respectively.

Figure 1: Risk Decomposition On The Right Side In (4.9)



On the other hand, the returns $E(r_b - r_f)/\sigma(r_b)$ and $E(R_b - r_f)/\sigma(R_b)$ are compensations for an investor to take the two unit of risks $(r_b - r_f)/\sigma(r_b)$ and $(R_b - r_f)/\sigma(R_b)$, separately. So the reward for bearing the risk $\rho(r_p, r_b)\cdot 1 + \rho(r_p, R_b)\cdot 1$ in $\sigma(r_p)$ explained by the two units of risks is the sum of two return projections: $E(r_b - r_f)/\sigma(r_b)$ and $E(R_b - r_f)/\sigma(R_b)$ onto the risk variable r_p as indicated in the Figure 2. The sum is exactly the right-side in (4.9). Thus, if an investor believes that only the explainable risk should be compensated, the right-side is a risk-adjusted return scaled by the explainable risk. Because the explainable risk is independent of the size of the asset risk $\sigma(r_p)$, the right-side is actually a risk-adjusted return of the asset r_p.

Figure 2: Return Decomposition on the right Side in (4.9)



Let's look at the term $S(r_b)$ before turning to the left-side in (4.9). We denote $\theta(r_b)=E([r_b-E(r_b)]^4)^{1/4}$ as the kurtosis of the portfolio returns[xii]. Since

$$\sigma((r_b-E(r_b))^2)=\sqrt{E((r_b-E(r_b))^4)-E((r_b-E(r_b))^2)^2}=\sqrt{\theta(r_b)^4-\sigma(r_b)^4},$$

We can express $S(r_b)$ in terms of the normalized skewness and kurtosis as

$$S(r_b)=\frac{\gamma(r_b)^3}{\sigma(r_b)\sigma((r_b-E(r_b))^2)}=\frac{\gamma(r_b)^3}{\sigma(r_b)\sqrt{\theta(r_b)^4-\sigma(r_b)^4}}=\frac{(\gamma(r_b)/\sigma(r_b))^3}{\sqrt{(\theta(r_b)/\sigma(r_b))^4-1}}.$$

Moreover, the two-fund money separation theorem (TFMS) implies that an investor invests the asset r_p in the risk-free asset r_f and the market portfolio r_b, that is, $r_p=\omega r_f+(1-\omega)r_b$, where ω is constant. So

$$\begin{aligned}S(r_p)&=\frac{(\gamma(\omega r_f+(1-\omega)r_b)/\sigma(\omega r_f+(1-\omega)r_b))^3}{\sqrt{(\theta(\omega r_f+(1-\omega)r_b)/\sigma(\omega r_f+(1-\omega)r_b))^4-1}}\\&=\frac{(\gamma((1-\omega)r_b)/\sigma((1-\omega)r_b))^3}{\sqrt{(\theta((1-\omega)r_b)/\sigma((1-\omega)r_b))^4-1}}\\&=\frac{((1-\omega)\gamma(r_p)/(1-\omega)\sigma(r_p))^3}{\sqrt{((1-\omega)\theta(r_b)/(1-\omega)\sigma(r_b))^4-1}}\\&=\frac{(\gamma(r_b)/\sigma(r_b))^3}{\sqrt{(\theta(r_b)/\sigma(r_b))^4-1}}\\&=S(r_b)\end{aligned}$$

After replacing $S(r_b)$ with $S(r_p)$, we are able to change the model in (4.9) to

$$\frac{E(r_p-r_f)(1+S(r_p))}{\sigma(r_p)}\approx\rho(r_p,r_b)E\left(\frac{r_b-r_f}{\sigma(r_b)}\right)+\rho(r_p,R_b)E\left(\frac{R_b-r_f}{\sigma(R_b)}\right)\qquad(4.10)$$

As we argue that the right-side in (4.9) is the risk-adjusted return for the investment r_p, the left-side should be a risk-adjusted return for the asset r_p. Thus, we can use the left-side in (4.10) as a RAPM. The RAPM has the following form:

$$z=\frac{E(r_p-r_f)}{\sigma(r_p)}\left(1+\frac{(\gamma(r_p)/\sigma(r_p))^3}{\sqrt{(\theta(r_p)/\sigma(r_p))^4-1}}\right)\qquad(4.11)$$

The RAPM in (3.4) is the same as the one in (4.11) if the skewness preference d in (3.4) is equal to

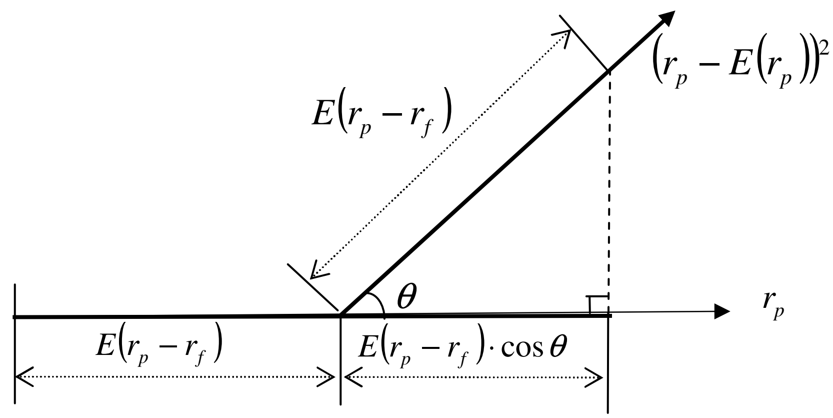
$$d = \frac{1}{\sqrt{\left(\theta(r_p)/\sigma(r_p)\right)^4 - 1}} \quad (4.12)$$

We can explain the term $S(r_p)$ in a geometric way. With the definitions in (4.4) and (4.5), we have

$$\begin{aligned} S(r_p) &= \frac{\gamma(r_p)^3}{\sigma(r_p)\sigma(R_p)} = \frac{E\left((r_p - E(r_p))^3\right)}{\sigma(r_p)\sigma(R_p)} = \frac{E\left((r_p - E(r_p)) \cdot R_p\right)}{\sigma(r_p)\sigma(R_p)} = \frac{Cor(r_p, R_p)}{\sigma(r_p)\sigma(R_p)} \\ &= \frac{\rho(r_p, R_b)\sigma(r_p)\sigma(R_p)}{\sigma(r_p)\sigma(R_p)} = \rho(r_p, R_p) = \rho\left(r_p, (r_p - E(r_p))^2\right) = \cos\theta \end{aligned}$$

Which implies that $S(r_p)$ is actually the correlation between the first-order return r_p and the second-order return $(r_p - E(r_p))^2$. If we use the Figure 3 to describe the return part in the RAPM, $S(r_p)$ is the value of the cosine function at θ. So the return part is the expected excess return plus the projection of the excess return along the second-order return onto the first-order return while the risk part in the RAPM is still the volatility $\sigma(r_p)$. The main factor to drive the skewness in the RAPM is the second-order return. When the angle θ between the first- and second-order returns is right, the skewness $\gamma(r_p)$ is zero. The RAPM becomes the traditional Sharpe Ratio. When the angle θ is acute (obtuse), the skewness $\gamma(r_p)$ is positive (negative). The RAPM is the Sharpe ratio adjusted upward (downward) by the amount $\cos\theta$ in the return part. In other words, the RAPM is the Sharpe ratio adjusted by the multiplier $(1+\cos\theta)$. The size of the multiplier is closely related to the position of the second-order return to the position of the first-order return. As a result, the RAPM is exactly the Sharpe ratio when the asset return doesn't show any skewness. It is upward (downward) adjusted when the return has a positive (negative) skewness. The RAPM is consistent with the fact that investors prefer a positive skewness to a negative skewness in an investment if other things are same.

Figure 3. The Return Decomposition in the RAPM in (4.11)



Also, the term $S(r_b)$ is closely related to market timing. Treynor and Mazuy (1966) define a market timing model as the regression:

$$r_{p,t} - r_f = \alpha_p + \beta_p\left(r_{b,t} - r_f\right) + \gamma_p\left(r_{b,t} - r_f\right)^2 + \varepsilon_t .$$

The coefficient γ_p on the expression $\left(r_{b,t} - r_f\right)^2$ is a measure of the asset's market timing. Since the second-order market return R_b is

$$R_b = \left(r_b - E(r_b)\right)^2 = \left(r_b - r_f\right)^2 - 2\left(r_b - r_f\right)\left(E(r_b) - r_f\right) + \left(E(r_b) - r_f\right)^2 ,$$

The expression $\beta\left(r_p, R_b\right)/\left(1 + S(r_b)\right)$ is the coefficient γ_p if we compare the model in (4.8) with the market timing model. So the expression containing market timing information.

When an investor has the expected utility in (3.1), i.e., the investor cares only the first three-moments of returns, the RAPM shouldn't contain the kurtosis. Why does the RAPM in (4.11) have the kurtosis? To answer the question, we first need to recall the assumptions in the extended RAPM theory. In the theory, we assume that all variables in the utility function are independent each other. So we regard the expected return $E\left(r_p\right)$, the volatility $\sigma\left(r_p\right)$, and the skewness $\gamma\left(r_p\right)$ as independent variables when deriving the RAPM in (3.4). The independence eliminates the opportunity to introduce the kurtosis. But we know from statistics that the moments of a distribution are not independent of each other. Roland and Xiang (2004) use this dependency to derive the asset pricing model in (4.1). Since the dependency introduces the kurtosis into the model, the RAPM certainly consists of the kurtosis if we use the mode to derive the RAPM.

It is also important to notice that the extended RAPM theory is held true under an individual equilibrium condition. We cannot model investors' different skewness preferences with the theory even if the investors have the same expected utility. Alternatively, we derive the RAPM in (4.11) by assuming that the TFMS is true. The conditions to ensure the TFMS are all investors have a hyperbolic absolute risk aversion utility and same return and risk preferences. These conditions also guarantee that the model in (4.1) is equivalent to a market equilibrium model. Obviously, the skewness preference d is the same for all investors under these conditions. In addition, the preference has the explicit expression in (4.12).

Lastly, we check the robustness of the RAPM in (4.11) using three properties of an expected utility. As we know, these properties are non-satiation, decreasing marginal utility and non-increasing absolute risk aversion. The motivation to use these properties is that we can treat the RAPM as an expected utility. Since each of the three conditions ($\partial z/\partial E(r_p) > 0$, $\partial z/\partial \sigma(r_p) < 0$, and $\partial z/\partial \gamma(r_p) > 0$) implies the corresponding property, we need to verify these conditions to see whether the RAPM has the properties.

For a calculation purpose, we rewrite the RAPM in (4.11) as

$$z = \frac{E(r_p) - r_f}{\sigma(r_p)}\left(1 + \frac{\gamma(r_p)^3}{\sigma(r_p)\sqrt{\theta(r_p)^4 - \sigma(r_p)^4}}\right) = \frac{E(r_p) - r_f}{\sigma(r_p)}(1 + \cos\theta) \qquad (4.13)$$

We assume that both $E(r_p) - r_f$ and $1 + \cos\theta$ are positive. We also assume that the skewness $\gamma(r_p)$ is non-zero. From (4.13), we can compute the derivatives with respect to the expected return and skewness to get

$$\frac{\partial z}{\partial E(r_p)} = \frac{1}{\sigma(r_p)}(1 + \cos\theta) > 0$$

And

$$\frac{\partial z}{\partial \gamma(r_p)} = \frac{E(r_p) - r_f}{\sigma(r_p)}\left(\frac{3\gamma(r_p)^2}{\sigma(r_p)\sqrt{\theta(r_p)^4 - \sigma(r_p)^4}}\right) > 0 .$$

Thus, the RAPM satisfies the first property of non-satiation and the third property of non-increasing absolute risk aversion. Similarly, we can compute the derivative of the RAPM in (4.13) with respect to the volatility as:

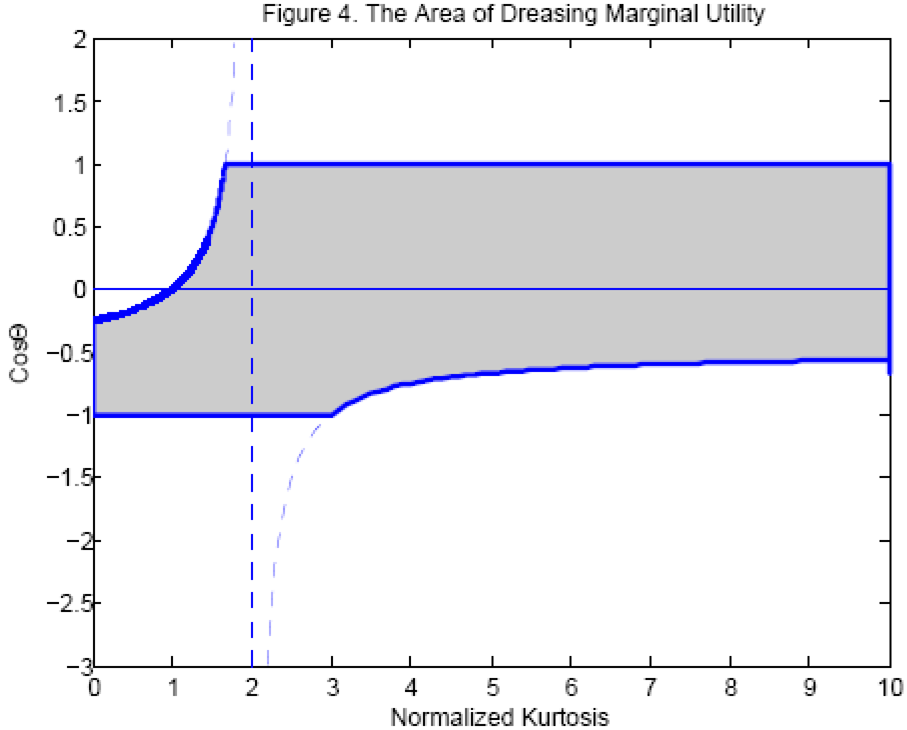
$$\frac{\partial z}{\partial \sigma(r_p)} = \frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-\frac{2\gamma(r_p)^3\left(\theta(r_p)^4-2\sigma(r_p)^4\right)}{\sigma(r_p)\left(\theta(r_p)^4-\sigma(r_p)^4\right)^{\frac{3}{2}}}\right)$$
$$= \frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-\frac{2\left(\theta(r_p)^4-2\sigma(r_p)^4\right)}{\theta(r_p)^4-\sigma(r_p)^4}\frac{\gamma(r_p)^3}{\sigma(r_p)\sqrt{\theta(r_p)^4-\sigma(r_p)^4}}\right)$$
$$= \frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-2\frac{\left(\theta(r_p)/\sigma(r_p)\right)^4-2}{\left(\theta(r_p)/\sigma(r_p)\right)^4-1}\cos\theta\right)$$

So, the RAPM has the second property of decreasing marginal utility once

$$-2\frac{\left(\theta(r_p)/\sigma(r_p)\right)^4-2}{\left(\theta(r_p)/\sigma(r_p)\right)^4-1}\cos\theta<1 \qquad (4.14)$$

The following Figure 1 shows the area where the condition in (4.14) holds. Particularly, the condition is true when $3/2 \le \left(\theta(r_p)/\sigma(r_p)\right)^4 \le 2$ and $\cos\theta < 1/2$, or $2 < \left(\theta(r_p)/\sigma(r_p)\right)^4$ and $\cos\theta \ge -1/2$.

Figure 4. The Area of Dreasing Marginal Utility



We test the condition in (4.14) using 299 indices. These indices include many asset classes such as equities, bonds, currencies, commodities, and hedge funds in different regions. The result shows that 290 indices satisfy the condition. The indices which violate the condition have the characteristics of a very small volatility, a large negative normalized skew, and a big normalized kurtosis. Six of them are hedged funds and three of them have negative excess returns. From the test, we believe with strong confidence that the RAPM can handle a majority of cases. However, we ensure the condition if reducing the skew in the RAPM by half. In this case, the RAPM becomes

$$z = \frac{E(r_p - r_f)}{\sigma(r_p)}\left(1 + \frac{(\gamma(r_p)/\sigma(r_p))^3}{2\sqrt{(\theta(r_p)/\sigma(r_p))^4 - 1}}\right).$$

We summarize what we did in this section. We used the asset pricing model to derive the three-moment RAPM under market equilibrium. The RAPM consists of the return mean, volatility, and skewness. It also includes kurtosis, but the kurtosis is for adjusting the skewness only. The RAPM is generally robust in terns of properties of an expected utility.

5. Conclusion

Although people use many forms of RAPMs, they still make efforts to look for new RAPMs. There are many reasons why they aren't satisfied with the existing RAPMs: the existing RAPMs don't have a link to an investor investment choice; they aren't suitable for evaluating an investor's strategy and investment product; they consist of only one risk measure; and most of them don't have a sound theory to support.

To address the weaknesses of the existing RAPMs, we extend Xiang, Liu and Wang (2012) (and Hooker and Xiang (2007))'s RAPM theory to include more than one risk measure. Similar to the original theory we developed, the extended theory links an investor's investment choice to a RAPM and specifies a function form of the RAPM. It demonstrates that an expected utility maximization is the same as the RAPM maximization under the individual equilibrium condition. Different from the original theory, the extended theory fails to identify all equivalent RAPMs. The lack of equivalent RAPMs offers us a freedom to choose one form of RAPM over another.

We used the extended theory to construct a three-moment RAPM. The RAPM is the Sharpe ratio when a return doesn't show any skewness. It is upward (or downward) adjusted when a return has a positive (or negative) skewness. We apply properties of an expected utility in the derivation of the RAPM. The properties are non-satiation, decreasing marginal utility and non-increasing absolute risk aversion. The individual RAPM has a skewness preference that is associated with each investor.

To better understand the skewness preference, we develop the three-moment RAPM by moving from the individual equilibrium to the market equilibrium. We use Harvey and Siddique's asset pricing model to derive the RAPM. The RAPM provides an explicit expression for the skewness preference. We also test the robustness of the RAPM with respect to properties of an expected utility.

In conclusion, this paper extends the RAPM literature to include multiple risks. We use a three-moment asset pricing model to construct a RAPM under individual equilibrium and a RAPM under market equilibrium with skewness. Our paper provides a theoretical justification to the relationships between investment choices, an asset pricing model, an individual RAPM, and a market RAPM. This research sheds light on the importance of selecting proper RAPMs and accurately measures the risk-adjusted performances. Future research will focus on building a specific measure of RAPM with skewness, i.e., the Sharpe ratio with skewness.

Endnotes

[i] Please refer to Ané and Geman (2000), and Chung, Johnson and Schill (2001) for some empirical work.

[ii] For the conditional version of three-moment models, please see Kraus and Litzenberger (1976), Friend and Westerfield (1980), and Ingersol (1990). For alternative three-moment models, please see Sears and Wei (1985), Nummelin (1994), Lim (1989), and Waldron (1990).

[iii] Roland and Xiang (2004) derive the same model following the principle: investors have the same rewards by taking the same level of risk. They don't use any expected utility in their theory.

[iv] This means that the investor can buy or sell assets in the investment set without any transaction cost.

[v] The investor doesn't change her utility function during the investment period.

[vi] If we assume that the wealth at the beginning of a period is one, we have $w = 1 + r_p$, where w and r_p corresponds to the wealth and return at the end of the period. Therefore, the expected utility can be a function of return, i.e., $E(U(w)) = E(U(1 + r_p))$.

[vii] If this is not true, there is an ambiguous situation such as the value of the utility function doesn't increase while the value of the measure strictly increases, which violates the definition of a RAPM.

[viii] Most of textbooks define $[E(r_p - E(r_p))]^3 / \sigma(r_p)^3$ as the skewness. However, we would like to call this expression as the normalized skewness.

[ix] $\partial z / \partial \gamma(r_p) > 0$ is the sufficient condition for the property of non-increasing absolute risk aversion.

[x] In theory, we can construct the two portfolios in such way that the first portfolio has the correlation $\rho(r_1, R_b) = 0$ and the second portfolio has the correlation $\rho(r_2, r_b) = 0$. For example, a market-neutral portfolio has a zero correlation with the market.

[xi] The three risk variables r_p, r_b, and R_b may not be on the same plane. However, we are interested in only the portion of r_p that can be explained by r_b and R_b. So we can assume that r_p, r_b, and R_b are on the same plane without loss of generality.

[xii] We define $[E(r_b - E(r_b))]^4 / \sigma(r_b)^4$ as the normalized kurtosis in this paper, instead of as the kurtosis called by most people.

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