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Optimized Formula for Distributing Coalition Worth in n-Person Fuzzy Cooperative Games

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In this paper, an optimized general formula for distributing gains of fuzzy cooperative players, fuzzy coalitions, is proposed. The game is characterized by a fuzzy characteristic function. The formula is derived from a linear multi-objective optimization model for fuzzy cooperative games developed earlier. In the developed model, players do not need to know precise information about the payoff value or even to form a crisp coalition. The optimality of the proposed formula is proved and the properties are verified. The formula is applied to a joint production problem for the purpose of validation and the obtained results are compared with Shapley value for solving fuzzy games. The proposed formula has the advantage of being easier and of less computational effort when compared to applying Shapley value.

1. Introduction

In classical cooperative game theory, players are not allowed to partially share their resources. Also, the coalition formation is based on an assumption that the players, at the very beginning of the negotiation process, know exactly the expected outcomes. But in realistic cooperative situations, the cooperating players are subjected to unpredictable worth (characteristic function) or payoff of the coalitions, which can be vaguely incorporated in the model.

In reality, it is highly difficult to arrive at such precise estimates where there are situations in which some players do not fully participate in a coalition, but to a certain extent. A coalition including players who participate partially is called a fuzzy coalition (Butnariu and Kroupa, 2008). The fuzzy coalition is a collection of players in which they share their resources where the membership grade shows to what extent a player participates in the formed coalition. On the other hand, players could only know vague information regarding the payoff of the formed coalition. This class of cooperative games is known as cooperative games with fuzzy coalitions and fuzzy characteristic functions or fuzzy games in short. Most researchers who have investigated the solutions of cooperative games focus their attention on crisp games or games with crisp coalitions and fuzzy characteristic function. Fuzzy games need more investigation.

In this paper, an optimization model is developed from which a general formula for distributing coalition worth among the players is derived and proved. The formula showed the same results as obtained from the developed model. Previous research work obtains

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the same results exploiting huge computational steps. As such, the developed formula needs less computational steps reducing computational effort.

The paper is organized as follows: In section 2, a literature review of related work on cooperative games either with fuzzy coalitions or fuzzy characteristic function or both is presented. Section 3, presents the developed model and the concluded optimized formula for distributing gains with its proof and properties. Section 4, gives an applicable example on a production management problem illustrating the application of the formula and comparing the results with the Hukuhara-Shapley function. Finally in section 5, the summary and conclusions are addressed.

2. Literature Review

The general question raised by any cooperative game can be described as follows: how should the utility (worth) available to a given coalition be used to allocate the individual payoff for each player in a fair way? Most researchers who have investigated the solutions of cooperative games focus their attention on crisp games or games with crisp coalitions and fuzzy characteristic function but cooperative games with fuzzy coalitions and fuzzy characteristic functions need more investigation.

(Shapley, 1953) introduces the Shapley Value function which is one of the most common solution concepts for crisp cooperative games. It shows a vector whose elements are players' share derived from several reasonable bases. (Butnariu, 1978) and (Butnariu, 2008) defined a Shapley value and showed the explicit form of the Shapley function but on a limited class of fuzzy games. Most games in the class are neither monotone non-decreasing nor continuous with regard to rates of players' participation although crisp games are often considered to be monotone non-decreasing. Thus, the class cannot be generalized.

(Tsurumi et al, 2001) defined new Shapley axioms and a new class of fuzzy games with Choquet integral form. This class of fuzzy games is both monotone non-decreasing and continuous with respect to players' participation. On the other hand, (Mares, 2000) and (Mares, 2001) were concerned about the uncertainty in the value of the characteristic function associated with a game. In their game models, the domain of the characteristic function of a game is still the class of crisp coalitions, but the coalition values allocated to players are fuzzy numbers.

(Borkotokey, 2008) undertook research on fuzzy games of which the coalition and the characteristic function are both fuzzy information. The difficulty in such an extension lied in proving that the extended game admits all the properties, as its counterpart in crisp sense. (Maali, 2009) introduced a linear programming model to solve crisp cooperative games. The solution is always Pareto optimal. It is based on the idea of the core but instead of requiring rationality for all groups, a multi-objective approach is developed considering the payoffs of the cooperated players in a given coalition as several objectives to be maximized simultaneously.

(Yu and Zhang, 2010) introduced a generalized form for fuzzy cooperative games. Based on the Hukuhara difference and the Choquet Integral, a game with a fuzzy coalitions and fuzzy characteristic function was studied, and the explicit form of a Shapley function was shown, called Hukuhara-Shapley function. However, this approach has a major limitation. It requires the existence of Hukuhara condition between fuzzy numbers, which does not always exist. In addition, computation of the Hukuhara-Shapley value requires a prohibitive number of calculations, particularly; it requires finding the crisp Shapley values first before solving for fuzzy coalitions, which increases the complexity of solution.

(Tharwat et al, 2011) developed a linear multi-objective optimization model for solving cooperative games with fuzzy coalitions and fuzzy characteristic function. The solution of the model is always pareto optimal and satisfying the properties of cooperative games. The idea is based on allocating the coalition worth to the players whose participation rate is larger than zero.

As mentioned before, previous research work focuses on crisp cooperative games while fuzzy games are less investigated. The developed model overcomes many difficulties in comparison with previous research. Three main deficiencies were concluded in which the model is capable overcoming them. Firstly, our developed model tackles cooperative games with fuzzy coalitions and fuzzy characteristic function. The class of fuzzy games with Choquet Integral form is studied, which proved to be the most successful one, admitting the standard features of the game. Secondly, the developed LMO/FCGames model solves fuzzy cooperative games with fuzzy characteristic function represented in a triangular fuzzy number without any conditions. Finally, the developed LMO/FCGames model is a simple optimization model. It is solved only for the given coalition(s) irrespective of the number of the cooperated players.

3. The Methodology and Model

The theory of cooperative games with vague cooperation is based on modeling fuzzy coalitions as fuzzy subsets of the set of all players who participate in the coalitions with some part of their power (participation rate). In the developed model, players do not need to know precise information about the payoff value or even to form a crisp coalition. The fuzzy set theory is then utilized to model the ambiguity of participation of each player in the formed coalitions and representing the characteristic function value (payoff) for these coalitions.

3.1 Model Formulation

Taking imprecision of information in decision making problems into account, a fuzzy characteristic function (i.e., fuzzy coalition value) can be incorporated into a cooperative game, represented by fuzzy numbers ω. Therefore, the characteristic function of such a game associates a crisp coalition with a fuzzy number ω(S). Assessing such fuzzy numbers for any crisp coalition S, a cooperative game with crisp coalitions and fuzzy characteristic function is defined by a pair (N, ω), where the fuzzy characteristic function $\omega: P(N) \rightarrow \mathfrak{R}^+ = \{\tilde{r} \in \mathfrak{R} | \tilde{r} \geq 0\}$ is such that $\omega(\varphi) = 0$, (where $\mathfrak{R}$ is the fuzzy real line).

A fuzzy cooperative game (FCGame), i.e. with fuzzy coalitions and fuzzy characteristic function, is a pair $(N, \tilde{\omega})$ where N is the set of players with $i \in N = \{1,\ldots,n\}$ and the fuzzy characteristic function $\tilde{\omega}: U \rightarrow \mathfrak{R}$ for all $U \subseteq N$, such that $\omega(\varphi) = 0$. A fuzzy coalition U is a fuzzy subset of N, which is a vector $U = \{U(1),\ldots,U(n)\}$ with coordinates U(i) contained in the interval [0,1]. The number U(i) indicates the membership grade of i in U. The class of all fuzzy subsets of U is denoted by F(N). For a fuzzy subset U, the α–level set is defined as: $[U]_\alpha = \{i \in N \mid U(i) \geq \alpha\}$, and the support set is denoted by $supp(U) = \{i \in N \mid U(i) > 0\}$.

Let x(U) = (x_1(U), … , x_n(U)) be the payoff vector, such that x_i(U) is the payoff of player i of the fuzzy coalition U ϵ F(N). This payoff vector is not a reasonable candidate for a solution unless it satisfies:

1. $X_i(U) = 0$, $\forall\ i \notin supp(U)$
2. Group rationality: $\sum_{i \in supp(U)} X_i(U) = \tilde{\omega}(U)$, $i = 1,\dots,n$
3. Individual rationality: $X_i(U) \geq [U(i) \,.\, \tilde{\omega}(\{i\})]$, $\forall\ i \in supp(U)$

Since, all players of coalition U cooperate together to maximize their profit, thus, a linear multi-objective programming model can be formulated for any fuzzy coalition U ϵ F(N) as follows (Tharwat et al, 2011):

Max $\left(\frac{x_1(U)}{C_1}, \frac{x_2(U)}{C_2}, \dots, \frac{x_n(U)}{C_n}\right)$
Subject to

$\sum_{i \in Supp(U)} x_i(U) = \tilde{\omega}(U)$, $i = 1,\dots,n$
$x_i(U) \geq [U(i) \,.\, \tilde{\omega}(\{i\})]$, $\forall\ i \in supp(U)$

$x_i(U) = 0$, $\forall\ i \notin supp(U)$

Where, $C_i = [\tilde{\omega}(U) - \tilde{\omega}(U - \{i\})] \,/\, \tilde{\omega}(U)$.

For simplicity we can write x_i instead of x_i(U), and the above formulated model can be transformed to the following optimization problem:

Max λ

Subject to

$\frac{1}{C_i} x_i \geq \lambda$, $i \in supp(U)$
$\sum_{i \in Supp(U)} x_i = \tilde{\omega}(U)$, $\forall\ i \in supp(U)$ Group Rationality Constraint
$x_i \geq U(i) \,.\, \tilde{\omega}(\{i\})$, $\forall\ i \in supp(U)$ Individual Rationality Constraint
$x_i = 0$, $\forall\ i \notin supp(U)$

Where, $C_i = [\tilde{\omega}(U) - \tilde{\omega}(U - \{i\})] \,/\, \tilde{\omega}(U)$.

In general, in order to identify the characteristic function of a game with fuzzy coalitions, it is often constructed on the basis of the characteristic function of the original crisp one. Extending the crisp game, the game with fuzzy coalitions can be represented by a mapping from the characteristic function of the crisp game to that of the game with fuzzy coalitions. Many extensions have been done, such as the (Tsurumi et al, 2001) extension, in which the characteristic function has the form:

$$t\tilde{\omega}(K) = \sum_{m=1}^{q(K)} \tilde{\omega}([K]_{r_m)}) \,.\, (r_m - r_{m-1})$$

Where, K ϵ F(N), Q(K) = {K(i) | K(i) > 0, i ϵ N}, q(K) is the cardinality of Q(K), i.e. q(K) = |Q(K)|, and r_m(K) = {i | i ϵ N, K(i) = r_m}. The elements in Q(K) are written in the increasing order as $r_1 < \ldots < r_{q(K)}$, and let $r_0 = 0$.

The above FCGame model is normally solved as an LP problem. In this paper we are seeking to propose a formula that could efficiently and directly solve the FCGame model. The proposed formula simply calculates the payoff distribution among the coalition players through transforming the payoff value of player i and the expected value of the fuzzy worth as follows:

$$x_i = \text{U(i) EV}_i + \frac{EV_U' \, [EV_U' - EV_{U-i}']}{\sum_{i=1}^{N}[EV_U' - EV_{U-i}']} \qquad \text{Eq 1}$$

Where, EV_i is the expected value for the fuzzy worth $\widetilde{\omega}$({i}) and $\text{EV}'_U = \text{EV}_U - \sum_{i \in Supp(U)} \text{U(i) EV}i$.

In the following sub-section, we will provide the derivation of Eq 1.

3.2 Model Derivation

To use the proposed formula to get the optimal solution for the developed fuzzy model, defuzzification of the fuzzy worth should be carried out. The expected value of fuzzy numbers method is used for the defuzzification process (Ren et al, 2016).

First, let us define the fuzzy set $a^{\sim}$ of a universe *X*. Its membership function $\mu_{\tilde{a}}$: *X* ⟶ [0,1]. The value $\mu_{\tilde{a}}$ (*x*) , *x* ∈ *X* represents the grade of membership of *x* in $a^{\sim}$.

A fuzzy number is a fuzzy set $a^{\sim}$ on the real line *R* with the membership function $\mu_{\tilde{a}}$ and it is defined as follows:

$$\mu_{\tilde{a}}(x) = \left\{ \begin{array}{ll} 0, & \forall\, x \in (-\infty, a_1] \\ f_a(x), & \forall\, x \in [a_1, a_2] \\ 1, & \forall\, x \in [a_2, a_3] \\ g_a(x), & \forall\, x \in [a_3, a_4] \\ 0, & \forall\, x \in [a_4, \infty) \end{array} \right\}$$

Where, the functions $f_a(x)$ on $[a_1, a_2]$ and $g_a(x)$ on $[a_3, a_4]$ are monotonically increasing and decreasing functions, respectively.

If these functions are linear, thus the fuzzy number $a^{\sim}$ is a trapezoidal fuzzy number and defined as $a^{\sim}$ = (a_1, a_2, a_3, a_4), and if $a_2 = a_3$, then $a^{\sim}$ is a triangular fuzzy number.

The α-level set of a fuzzy number can be defined as an ordinary set $a_\alpha = \{x \epsilon R \mid \mu_{\tilde{a}}(x) \geq \alpha\}$ for α ϵ [0,1]. The α-level set is a closed and bounded interval, defined as $a_\alpha = [f_a^{-1}(\alpha), g_a^{-1}(\alpha)]$ where, $f_a^{-1}(\alpha) = \inf\{x: \mu_{\tilde{a}}(x) \geq \alpha\}$ and $g_a^{-1}(\alpha) = \sup\{x: \mu_{\tilde{a}}(x) \geq \alpha\}$.

Based on (Heilpern, 1992), the expected interval of a fuzzy number $a^{\sim}$, denoted by EI($a^{\sim}$), can be defined as follows:

$$\mathrm{EI}(a^{\sim}) = [E_1^a, E_2^a] = [\int_0^1 f_a^{-1}(\alpha), \int_0^1 g_a^{-1}(\alpha)]$$

The expected value of a fuzzy number $a^{\sim}$, denoted by EV($a^{\sim}$), is defined as the half point of its expected interval:

$$\mathrm{EV}(a^{\sim}) = \frac{E_1^a + E_2^a}{2}$$

Thus the expected value for the fuzzy worth $\widetilde{\omega}$(U) is $\mathrm{EV}_U = \frac{E_1^U + E_2^U}{2}$

Second, we will prove that the proposed formula is optimal for the developed fuzzy model and its properties are investigated.

Lemma 1:

Following Abbasi (2012), for the optimization model:

Max Min

$$(a_1 = f_1(x_1), a_2 = f_2(x_2), \dots, a_N = f_N(x_N))$$

Subject to

$$\sum_{i=1}^{N} x_i = T$$
$$x_i \geq 0$$

Where $f_i(x_i)$; i = 1, 2, …, N is an increasing function and ≥ 0, the optimum solution is obtained by solving the following two equations:

$$a_1 = a_2 = \dots = a_N = a$$
$$\sum_{i=1}^{N} x_i = T$$

Through applying lemma 1 to the FCGame model, the following transformation on x_i could be applied as

$x_i' = x_i - U(i)\,EV_i$, i = 1, 2, …, N

Therefore, substituting for x_i in the FCGame model, we have:

Individual Rationality Constraint

$x_i' + U(i)\,EV_i \geq U(i)EV_i$

Thus, $x_i' \geq 0$

Group Rationality Constraint

$\sum_{i \in Supp(U)} [x_i' + \mathrm{U(i)\ EV}i] = EV_U$

$\sum_{i \in Supp(U)} x_i'$ = EV$_U$ - $\sum_{i \in Supp(U)}$ U(i) EVi = EV_U'

Thus, $\sum_{i \in Supp(U)} x_i' = EV_U'$

Since the optimization model considers the players of coalition U, therefore the last constraint is redundant and the FCGame model becomes:

Max λ

Subject to

$\frac{1}{C_i'} x_i' \geq \lambda$, i ϵ supp(U)

$\sum_{i \in Supp(U)} x_i' = EV_U'$, ∀ i ϵ supp(U)

$x_i' \geq 0$ ∀ i ϵ supp(U)

Which can be written as:

Max Min $\left(\frac{x_1'}{C_1'}, \frac{x_2'}{C_2'}, \ldots, \frac{x_N'}{C_N'}\right)$

Subject to

$\sum_{i=1}^{N} x_i' = EV_U'$,

$x_i' \geq 0$ i = 1,2,…,N

Based on Lemma 1, the optimal solution of the last formulated model is solving the following two equations:

$$\frac{x_1'}{C_1'} = \frac{x_2'}{C_2'} = \cdots = \frac{x_N'}{C_N'} = a$$

$$\sum_{i=1}^{N} x_i' = EV_U'$$

By replacing $x_i' = aC_i'$

$$a\sum_{i=1}^{N} C_i' = EV_U'$$

$$\frac{x_i'}{C_i'} \sum_{i=1}^{N} C_i' = EV_U'$$

$$x_i' = \frac{C_i' \, EV_U'}{\sum_{i=1}^{N} C_i'}$$

$$\therefore x_i - U(i)\, EV_i = \frac{C_i' \, EV_U'}{\sum_{i=1}^{N} C_i'}$$

$$x_i = U(i)\, EV_i + \frac{C_i' \, EV_U'}{\sum_{i=1}^{N} C_i'}$$

$$C_i' = \frac{[EV_U' - EV_{U-i}']}{EV_U'}$$

$$\therefore x_i = U(i)\, EV_i + \frac{EV_U' \, [EV_U' - EV_{U-i}']}{\sum_{i=1}^{N} [EV_U' - EV_{U-i}']}$$

This completes the derivation of Eq 1 proposed earlier.

The proposed model satisfies the following properties related to cooperative games:

1. The payoffs of players obtained by solving the concluded formula satisfy both the individual and the group rationality.
2. Zero player (Null player): zero player receives zero gain.

In the next section, a numerical example in production management is given to illustrate the application of the concluded formula and to show its advantage over using the Hukuhara-Shapley value.

4. Model Implementation

We will adapt the case study presented by (Yu and Zhang, 2010). The case study considers a joint production model in which three decision makers pool three resources to make seven finished products. The three decision makers (referred to as DM1, DM2 and DM3) possess three different initial resources. Decision maker i has 10 tons of resource R_i and can produce n_i tons of product P_{ii}, where i = 1, 2, 3. Let us consider that the decision makers decide to undertake a joint product: if decision makers i and j cooperate, they will produce n_{ij} tons of product P_{ij}, and if all three cooperate, n_{123} tons of product P_{123} can be produced. The effective output of each finished product is as shown in the following table

The effective output and the average profit of each finished product

	Product			
	P11	**P22**	**P33**	**P12**
Product Output (tons)	8	9	10	18
Average Profit ($1000)	(1.8,2.0,2.2)	(2.9,3.0,3.1)	(0.9,1.0,1.2)	(2.9,3.1,3.3)
	P13	**P23**	**P123**	
Product Output (tons)	17.5	18	28	
Average Profit ($1000)	(2.0,2.3,2.6)	(3.0,3.2,3.4)	(3.2,3.5,3.8)	

It is natural for the three decision makers to try to evaluate the revenue of the joint project in order to decide whether the project can be realized or not. However, the average profit per ton of each product is dependent on a number of factors such as product market price, product cost, consumer demand, the relation of commodity supply and demand, etc. Hence, the average profit of each product is an approximate evaluation, which is represented by a triangular fuzzy numbers as shown in the table.

During the early period of the joint project, every decision maker has to consider how many resources he or she should provide in the cooperation (partial participation). As we all know, each decision maker does not need to supply all of his resources to cooperate in real life; it depends on individual preference.

Suppose decision maker 1 would cooperate with decision maker 2 and 3, i.e. U = {1,2,3}. In this coalition, DM 1 would supply 6 tons of resource R_1 (i.e., participation rate of DM 1 is 0.6), DM 2 would supply 2 tons of resource R_2 (i.e., participation rate of DM 2 is 0.2), and DM 3 would supply 10 tons of resource R_3 (i.e., participation rate of DM 3 is 1). Now, it is required to evaluate the following scenario:

Give an allocation of the decision makers' profit share in the fuzzy coalition, U = {1, 2, 3}, where they cooperate together introducing different resources participation rates.

The fuzzy worth of each of the crisp coalition could be calculated as follows:

$\omega\{1\} = (14.4, 16, 17.6)$ $\omega\{2\} = (26.1, 27, 27.9)$ $\omega\{3\} = (9,10,12)$
$\omega\{1,2\} = (52.2, 55.8, 59.4)$
$\omega\{1,3\} = (35, 40.25, 45.5)$
$\omega\{2,3\} = (54, 57.6, 61.2)$
$\omega\{1,2,3\} = (89.6, 98, 106.4)$

Calculating Tsurumi et al.'s extension values for these fuzzy coalitions, we get:

$t\omega\{1,2\} = (16.8, 17.6, 18.3)$
$t\omega\{1,3\} = (25.1, 28.3, 31.4)$
$t\omega\{2,3\} = (19.0 ,19.7, 20.4)$
$t\omega\{1,2,3\} = (36.02, 39.8, 43.58)$

Defuzzifying these values based on (Heilpern, 1992) and applying our proposed formula, an assessment of the payoff of each decision maker in the given fuzzy coalition could be evaluated as follows:

X_1 = \$14.93 X_2 = \$8.53 X_3 = \$16.35

On the other hand, using the Hukuhara-Shapley function and Defuzzifying the resulted values, the decision makers' payoff would be as follows:

X_1 = \$14.98 X_2 = \$8.56 X_3 = \$16.16

It is obvious that our formula yields almost same results as the Hukuhara-Shapley function but with less computational effort (no need to calculate the crisp Shapley values first) and overcoming the Hukuhara difference condition between fuzzy numbers.

5. Summary and Conclusions

The uncertainty and vagueness of expectations are of major interests to natural components of cooperative situations, in general. Thus, that have given rise to rise to several kinds of fuzzy games. In this paper, the games with fuzzy coalitions and fuzzy characteristic functions are considered. An optimized formula for distributing coalition worth in n-person fuzzy cooperative games was proposed. It was shown and proved that the concluded formula is an imputation and satisfying several desired properties and reducing many computational effort. It overcomes the major limitation of Hukuhara-Shapley function as it does not require any conditions between triangular fuzzy numbers of the characteristic function.

A numerical example in a production management problem was applied to demonstrate the application of the proposed formula and to compare the solution of the proposed formula to that of Hukuhara-Shapley function. The results showed that although the proposed formula provides almost same results as the Hukuhara-Shapley function, but it direct to apply with less computational effort and overcoming the condition of the Hukuhara difference. The results showed that although the presented formula provides almost identical results as the Shapely values, it is easier to use and apply.

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